Physics 742 – Graduate Quantum Mechanics 2

First Exam, Spring 2019

Please note that some possibly helpful formulas are listed below or on the handout. Each question is worth twenty points.

1. A particle of mass m in one dimension is in the potential $V(x)=\alpha\sqrt{|x|}$. Using the WKB method, estimate the energy of the n'th eigenstate. ***Hint***: I found it useful to define $x=y^2$ and $z=E-\alpha y$.

2. A particle of mass m in one dimension is in the potential $V(x)=\alpha\sqrt{|x|}$. Using the variational principle with trial wave function $\psi(x)=e^{-\lambda|x|/2}$, estimate the energy of the ground state. I recommend using $\langle\psi|P^2|\psi\rangle=|P|\psi\rangle|^2$ when estimating the kinetic term.

3. A particle of mass m in two dimensions is in the potential $V(x)=\frac{1}{2}m\omega^2(X^2+Y^2)+\delta X^2Y^2$, where δ is small. Name and find the energies of the eigenstates of the unperturbed Hamiltonian in the limit $\delta=0$. Find the ground state eigenstate to first order in δ, and its energy to second order in δ.

4. An electron is in a three-dimensional harmonic oscillator with Coulomb potential $V_c(r)=\frac{1}{2}m\omega^2r^2$.
   (a) Write the spin-orbit coupling in terms of $\mathbf{L}^2$, $\mathbf{S}^2$, and $\mathbf{J}^2$, where $\mathbf{J}=\mathbf{L}+\mathbf{S}$.
   (b) For l = 0, what are the eigenvalues or possible eigenvalues of $\mathbf{L}^2$, $\mathbf{S}^2$, and $\mathbf{J}^2$? Argue that for states with l = 0, the spin-orbit coupling causes no shift in energy.
   (c) For l = 1, what are the eigenvalues or possible eigenvalues of $\mathbf{L}^2$, $\mathbf{S}^2$, and $\mathbf{J}^2$? Find the corresponding shift in energies.

5. A particle of mass μ and wave number k moving in the $+z$ direction scatters from a potential $V=V_0xye^{-\alpha\mathbf{r}^2/2}$, where V_0 is small. Find the differential and total cross-section in the first Born approximation. For the total cross-section, you may leave one integral uncompleted.

Possibly Helpful Formulas

Spin-Orbit Coupling

$$W_{\text{SO}}=\frac{g}{4m^2c^2}\frac{1}{r}\frac{dV_c(r)}{dr}\mathbf{L}\cdot\mathbf{S}$$

Born Approximation

$$\frac{d\sigma}{d\Omega}=\frac{\mu^2}{4\pi^2\hbar^4}\left|\int d^3\mathbf{r}V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2$$

$$\mathbf{K}^2=2k^2(1-\cos\theta)$$

1D Harmonic Oscillator

$$X=\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)$$

$$a|n\rangle=\sqrt{n}|n-1\rangle$$

$$a^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$$

Possibly Helpful Integrals

$$\int_0^\infty x^n e^{-Ax}dx = \begin{cases} n!/A^{n+1} & n \text{ integer,} \\ \Gamma(n+1)/A^{n+1} & \text{all } n. \end{cases} \qquad \int_{-\infty}^{\infty} e^{-Ax^2/2+Bx}dx = \sqrt{\frac{2\pi}{A}}e^{B^2/2A},$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}. \qquad \int_{-\infty}^{\infty} xe^{-Ax^2/2+Bx}dx = \frac{B}{A}\sqrt{\frac{2\pi}{A}}e^{B^2/2A}.$$

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