Physics 742 – Graduate Quantum Mechanics 2

Midterm Exam, Spring 2018

Please note that some possibly helpful formulas and integrals appear on the second page. Each question is worth twenty points.

1. A quantum system in the state $|\psi\rangle$ is measured using the operator *A*, where in some basis,

$$|\psi\rangle = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix} \quad \text{and} \quad A = a\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}.$$

What are the possible results that could occur, and what are their corresponding probabilities? In each case, what is the state vector after the measurement in this basis?

2. Two particles of mass *m* lies in one-dimensional coupled harmonic oscillator with potential $V(X_1, X_2) = \frac{1}{2}m\omega_0^2\left(2X_1^2 - 2\sqrt{2}X_1X_2 + 3X_2^2\right)$. Find the energy of all eigenstates.

3. A particle of mass *m* lies in a two-dimensional symmetric harmonic oscillator with classical frequency ω. It is placed in a two-dimensional coherent state labeled by two complex numbers *z* and *w*, so that the normalized state $|z,w\rangle$ satisfies

$$a_x|z,w\rangle = z|z,w\rangle, \quad a_y|z,w\rangle = w|z,w\rangle,$$

where a_x and a_y are the lowering operators in the *x*- and *y*-direction respectively. Find the expectation value for this state $|z,w\rangle$ for the angular momentum operator $L_z = XP_y - YP_x$.

4. Two identical non-interacting spinless particles are placed in a 1D infinite square well with allowed region $0 < x < a$. One of them is in the ground state ($n = 1$) and the other in the first excited state ($n = 2$).
   (a) Find the wave function for the two particles $\psi(x_1, x_2)$ if they are (i) distinguishable, (ii) bosons, or (iii) fermions.
   (b) In each case, find the probability density that they are both at $x = \frac{1}{3}a$.

5. An electron with its spin up along an axis in the *xy*-plane at an angle ϕ compared to the xz-axis has normalized state vector given by $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\e^{i\phi}\end{pmatrix}$. Suppose this electron is at an angle randomly chosen from $\phi \in \left\{-\frac{1}{3}\pi, 0, \frac{1}{3}\pi\right\}$, each choice equally probable.
   (a) What is the state operator ρ written as a 2×2 matrix?
   (b) What would be the expectation values of each of the spin operators, given by

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Possibly Helpful Formulas

Coherent States:
$a|z\rangle = z|z\rangle, \quad \langle z|a^\dagger = \langle z|z^*$

1D Harmonic Oscillator

$$P = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right)$$

$$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Infinite Square Well
Allowed region $0 < x < a$

$$\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right)$$

$$E_n = \frac{\pi^2\hbar^2 n^2}{2ma^2}$$

Coupled H.O.:

$$V = \tfrac{1}{2}\sum_{i,j} K_{ij} X_i X_j$$

$$\omega_i = \sqrt{k_i/m}$$

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