Physics 742 – Graduate Quantum Mechanics 2

Midterm Exam, Spring 2018

Please note that some possibly helpful formulas and integrals appear on the second page. Each question is worth twenty points.

1. A quantum system in the state $|\psi\rangle$ is measured using the operator *A*, where in some basis,

$$|\psi\rangle = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix} \quad \text{and} \quad A = a\left(\begin{array}{cc|c}0 & 1 & 0\\1 & 0 & 0\\ \hline 0 & 0 & 1\end{array}\right).$$

What are the possible results that could occur, and what are their corresponding probabilities? In each case, what is the state vector after the measurement in this basis?

We first need to find the eigenvalues of the matrix *A*. Since *A* is block diagonal, as marked above, we can see that one eigenvalue simply has its eigenstate in the third position, with eigenvalue *a*. To find the other two eigenstates and eigenvalues, we need to find the eigentates of the matrix $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$. This comes up so often that we know the eigenvectors; they are $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm 1\end{pmatrix}$, which have eigenvalues of ± 1. We multiply this by *a* to get the eigenvalues, and putting it back into three-component column matrix. Our three eigenvectors are therefore

$$|a,1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \quad |-a\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \quad |a,2\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

I have labeled the eigenstates by their eigenvalues, with an additional label when needed for the degenerate eigenvalues.

The possible outcomes for the measurements are simply $\pm a$, with corresponding probabilities

$$P(+a) = \sum_n \left|\langle +a,n|\psi\rangle\right|^2 = \left|\langle a,1|\psi\rangle\right|^2 + \left|\langle a,2|\psi\rangle\right|^2 = \left|\frac{1}{3\sqrt{2}}\begin{pmatrix}1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\2\\2\end{pmatrix}\right|^2 + \left|\frac{1}{3}\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\2\\2\end{pmatrix}\right|^2$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{1}{2} + \frac{4}{9} = \frac{17}{18},$$

$$P(-a) = \left|\langle -a|\psi\rangle\right|^2 = \left|\frac{1}{3\sqrt{2}}\begin{pmatrix}1 & -1 & 0\end{pmatrix}\begin{pmatrix}1\\2\\2\end{pmatrix}\right|^2 = \left(\frac{-1}{3\sqrt{2}}\right)^2 = \frac{1}{18}.$$

The state afterwards will depend on which case we are in, so we have

$$\left|\psi^+\right\rangle_{+a} = \frac{1}{\sqrt{P(a)}}\sum_n |a,n\rangle\langle a,n|\psi\rangle = \sqrt{\frac{18}{17}}\left(|a,1\rangle\langle a,1|\psi\rangle + |a,2\rangle\langle a,2|\psi\rangle\right)$$

$$= \sqrt{\frac{18}{17}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}\frac{1}{\sqrt{2}} + \begin{pmatrix}0\\0\\1\end{pmatrix}\frac{2}{3}\right] = \sqrt{\frac{18}{17}}\frac{1}{6}\begin{pmatrix}3\\3\\4\end{pmatrix} = \frac{1}{\sqrt{34}}\begin{pmatrix}3\\3\\4\end{pmatrix},$$

$$\left|\psi^+\right\rangle_{-a} = \frac{1}{\sqrt{P(a)}}|a\rangle\langle a|\psi\rangle = \sqrt{18}\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}\left(\frac{-1}{3\sqrt{2}}\right) = -\sqrt{\frac{18}{36}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}.$$

2. **Two particles of mass *m* lies in one-dimensional coupled harmonic oscillator with potential $V(X_1, X_2) = \frac{1}{2}m\omega_0^2\left(2X_1^2 - 2\sqrt{2}X_1X_2 + 3X_2^2\right)$. Find the energy of all eigenstates.**

We first write the interaction in the form $V\left(X_i, X_j\right) = \frac{1}{2}\sum_{i,j} K_{ij} X_i X_j$, where K_{ij} is symmetric. Remembering that the cross-term has to be split between K_{12} and K_{21}, we can write *K* as a two by two matrix:

$$K = m\omega_0^2\begin{pmatrix}2 & -\sqrt{2}\\ -\sqrt{2} & 3\end{pmatrix}.$$

We then need to find the eigenvalues of *K*. If we factor out the common factor of $m\omega_0^2$, the remaining matrix's eigenvalues can be found by solving the characteristic equation:

$$0 = \det\left(\hat{K} - \lambda\mathbf{1}\right) = \begin{pmatrix}2-\lambda & -\sqrt{2}\\ -\sqrt{2} & 3-\lambda\end{pmatrix} = (2-\lambda)(3-\lambda) - \left(-\sqrt{2}\right)^2 = 6 - 5\lambda + \lambda^2 - 2$$

$$= \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1).$$

The roots of this equation are obviously $\lambda = 1$ and $\lambda = 4$, which means that the eigenvalues of *K* are $k_1 = m\omega_0^2$ and $k_2 = 4m\omega_0^2$. We then find the classical frequencies, which are

$$\omega_1 = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{m\omega_0^2}{m}} = \omega_0, \quad \omega_2 = \sqrt{\frac{k_2}{m}} = \sqrt{\frac{4m\omega_0^2}{m}} = 2\omega_0.$$

The energies corresponding to these frequencies take the from $E = \hbar\omega_i\left(n_i + \frac{1}{2}\right)$, so if we call our states $\left|n_1, n_2\right\rangle$, the corresponding energies will be the sum of these two contributions, or

$$E_{n_1,n_2} = \hbar\omega_1\left(n_1 + \tfrac{1}{2}\right) + \hbar\omega_2\left(n_2 + \tfrac{1}{2}\right) = \hbar\omega_0\left(n_1 + \tfrac{1}{2} + 2n_2 + \tfrac{2}{2}\right) = \hbar\omega_0\left(n_1 + 2n_2 + \tfrac{3}{2}\right).$$

3. **A particle of mass *m* lies in a two-dimensional symmetric harmonic oscillator with classical frequency ω. It is placed in a two-dimensional coherent state labeled by two complex numbers *z* and *w*, so that the normalized state $|z,w\rangle$ satisfies**

$$a_x|z,w\rangle = z|z,w\rangle, \quad a_y|z,w\rangle = w|z,w\rangle,$$

where a_x and a_y are the lowering operators in the *x*- and *y*-direction respectively. Find the expectation value for this state $|z,w\rangle$ for the angular momentum operator $L_z = XP_y - YP_x$.

We first note that we can get two addition pieces of information by taking the Hermitian conjugate of these relations, namely

$$\langle z,w|a_x^\dagger = \langle z,w|z^*, \quad \langle z,w|a_y^\dagger = \langle z,w|w^*.$$

We now write the angular momentum operator in terms of raising and lowering operators, which will yield

$$\begin{aligned} L_z = XP_y - YP_x &= \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}}\left[\left(a_x + a_x^\dagger\right)i\left(a_y^\dagger - a_y\right) - \left(a_y + a_y^\dagger\right)i\left(a_x^\dagger - a_x\right)\right] \\ &= \tfrac{1}{2}i\hbar\left[\left(a_x a_y^\dagger - a_x a_y + a_x^\dagger a_y^\dagger - a_x^\dagger a_y\right) - \left(a_y a_x^\dagger - a_y a_x + a_y^\dagger a_x^\dagger - a_y^\dagger a_x\right)\right] \\ &= \tfrac{1}{2}i\hbar\left[2a_x a_y^\dagger - 2a_x^\dagger a_y\right] = i\hbar\left(a_y^\dagger a_x - a_x^\dagger a_y\right). \end{aligned}$$

We note that we were assisted in this simplification by the fact that every pair of operators that were multiplied always commute. We now simply compute the expectation value by always letting the lowering operators act to the right, and the lowering operators to the right, to yield

$$\begin{aligned} \langle z,w|L_z|z,w\rangle &= i\hbar\left(\langle z,w|a_y^\dagger a_x|z,w\rangle - \langle z,w|a_x^\dagger a_y|z,w\rangle\right) = i\hbar\left(w^* z\langle z,w|z,w\rangle - z^* w\langle z,w|z,w\rangle\right) \\ &= i\hbar\left(w^* z - z^* w\right) = 2\hbar\,\mathrm{Im}\left(z^* w\right). \end{aligned}$$

Once we had simplified, it, that wasn't too bad!

4. **Two identical non-interacting spinless particles are placed in a 1D infinite square well with allowed region 0 < *x* < *a*. One of them is in the ground state (*n* = 1) and the other in the first excited state (*n* = 2).**

(a) Find the wave function for the two particles $\psi(x_1, x_2)$ if they are (i) distinguishable, (ii) bosons, or (iii) fermions.

For non-interacting distinguishable particles, they can simply be put in their two states, with state vector $|\phi_1, \phi_2\rangle$. The resulting wave function is

$$\psi_D(x_1, x_2) = \phi_1(x_1)\phi_2(x_2) = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right).$$

For bosons, the state vector must be symmetrized, so we are in the state $\frac{1}{\sqrt{2}}\left(|\phi_1, \phi_2\rangle + |\phi_2, \phi_1\rangle\right)$. For fermions, they must be anti-symmetrized, so we are in the state $\frac{1}{\sqrt{2}}\left(|\phi_1, \phi_2\rangle - |\phi_2, \phi_1\rangle\right)$. The corresponding wave functions in each case are

$$\psi_B(x_1, x_2) = \frac{\phi_1(x_1)\phi_2(x_2) + \phi_2(x_1)\phi_1(x_2)}{\sqrt{2}} = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right],$$

$$\psi_F(x_1, x_2) = \frac{\phi_1(x_1)\phi_2(x_2) - \phi_2(x_1)\phi_1(x_2)}{\sqrt{2}} = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right],$$

(b) In each case, find the probability density that they are both at $x = \frac{1}{3}a$.

The probability density is just the wave function at this point squared, which is

$$\left|\psi_D\left(\tfrac{1}{3}a, \tfrac{1}{3}a\right)\right|^2 = \left|\frac{2}{a}\left[\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)\right]\right|^2 = \left|\frac{2}{a}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right|^2 = \left|\frac{3}{2a}\right|^2 = \frac{9}{4a^2},$$

$$\left|\psi_B\left(\tfrac{1}{3}a, \tfrac{1}{3}a\right)\right|^2 = \left|\frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\right]\right|^2$$

$$= \left|\frac{\sqrt{2}}{a}\left[\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right]\right|^2 = \left|\frac{\sqrt{2}}{a}\cdot\frac{3}{2}\right|^2 = \frac{9}{2a^2},$$

$$\left|\psi_F\left(\tfrac{1}{3}a, \tfrac{1}{3}a\right)\right|^2 = \left|\frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\right]\right|^2 = \left|\frac{\sqrt{2}}{a}\left[\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right]\right|^2$$

$$= \left|\frac{\sqrt{2}}{a}\cdot 0\right|^2 = 0.$$

5. An electron with its spin up along an axis in the *xy*-plane at an angle ϕ compared to the x-axis has normalized state vector given by $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ e^{i\phi}\end{pmatrix}$. Suppose this electron is at an angle randomly chosen from $\phi \in \{-\tfrac{1}{3}\pi, 0, \tfrac{1}{3}\pi\}$, each choice equally probable.

(a) What is the state operator ρ written as a 2×2 matrix?

The state operator is given by

$$\rho = \sum_i f_i |\psi_i\rangle\langle\psi_i| = \frac{1}{3}\cdot\left(\frac{1}{\sqrt{2}}\right)^2 \left[\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} + \begin{pmatrix}1\\ e^{\frac{1}{3}i\pi}\end{pmatrix}\begin{pmatrix}1 & e^{-\frac{1}{3}i\pi}\end{pmatrix} + \begin{pmatrix}1\\ e^{-\frac{1}{3}i\pi}\end{pmatrix}\begin{pmatrix}1 & e^{\frac{1}{3}i\pi}\end{pmatrix}\right]$$

$$= \frac{1}{6}\left[\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \begin{pmatrix}1 & e^{-\frac{1}{3}i\pi}\\ e^{\frac{1}{3}i\pi} & 1\end{pmatrix} + \begin{pmatrix}1 & e^{\frac{1}{3}i\pi}\\ e^{-\frac{1}{3}i\pi} & 1\end{pmatrix}\right] = \frac{1}{6}\begin{pmatrix}3 & 1+e^{-\frac{1}{3}i\pi}+e^{\frac{1}{3}i\pi}\\ 1+e^{\frac{1}{3}i\pi}+e^{-\frac{1}{3}i\pi} & 3\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}3 & 1+2\cos\left(\frac{1}{3}\pi\right)\\ 1+2\cos\left(\frac{1}{3}\pi\right) & 3\end{pmatrix} = \frac{1}{6}\begin{pmatrix}3 & 2\\ 2 & 3\end{pmatrix}.$$

(b) What would be the expectation values of each of the spin operators, given by

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The expectation value for any operator is $\langle A\rangle = \mathrm{Tr}(\rho A)$, so we have

$$\langle S_x\rangle = \mathrm{Tr}(\rho S_x) = \frac{\hbar}{6\cdot 2}\mathrm{Tr}\left[\begin{pmatrix}3 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = \frac{\hbar}{12}\mathrm{Tr}\begin{pmatrix}2 & 3\\ 3 & 2\end{pmatrix} = \frac{4\hbar}{12} = \frac{\hbar}{3},$$

$$\langle S_y\rangle = \mathrm{Tr}(\rho S_x) = \frac{\hbar}{6\cdot 2}\mathrm{Tr}\left[\begin{pmatrix}3 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] = \frac{\hbar}{12}\mathrm{Tr}\begin{pmatrix}2i & -3i\\ 3i & -2i\end{pmatrix} = 0,$$

$$\langle S_y\rangle = \mathrm{Tr}(\rho S_x) = \frac{\hbar}{6\cdot 2}\mathrm{Tr}\left[\begin{pmatrix}3 & 2\\ 2 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] = \frac{\hbar}{12}\mathrm{Tr}\begin{pmatrix}3 & -2\\ 2 & -3\end{pmatrix} = 0.$$

So we're done!

Possibly Helpful Formulas

Coherent States:
$a|z\rangle = z|z\rangle,\quad \langle z|a^\dagger = \langle z|z^*$

1D Harmonic Oscillator

$$P = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right)$$

$$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Infinite Square Well
Allowed region $0 < x < a$

$$\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right)$$

$$E_n = \frac{\pi^2\hbar^2 n^2}{2ma^2}$$

Coupled H.O.:

$$V = \tfrac{1}{2}\sum_{i,j} K_{ij} X_i X_j$$

$$\omega_i = \sqrt{k_i/m}$$