Physics 741 – Graduate Quantum Mechanics 1

Final Exam, Fall 2018

Each question is worth 25 points, with points for each part marked separately. Some possibly useful formulas can be found at the end of the exam.

1. A hydrogen atom is in the state $|\psi\rangle = \sqrt{\tfrac{1}{3}}\left|2,1,0,\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|2,1,1,-\tfrac{1}{2}\right\rangle$, where we are using the notation $|n,l,m,m_s\rangle$, where l, m, and m_s correspond to L^2, L_z, and S_z respectively.
   (a) [5] If one were to measure the operators L_z, and S_z, what would be the possible outcomes and corresponding probabilities?
   (b) [20] If one were to measure J^2 and J_z, what would be the possible outcomes and corresponding probabilities?

2. An electron is in a region with electric field $\mathbf{E} = E_0\hat{\mathbf{x}}$ and magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$.
   (a) [8] Find an electrostatic potential U that alone can account for this electric field. Which coordinate(s) is it independent of? Find a vector potential A that is independent of the same coordinate(s) and accounts for the magnetic field.
   (b) [9] Write the Hamiltonian explicitly. Find three operators that commute with the Hamiltonian and with each other. These operators might be spin operators, momentum operators, or angular momentum operators. Give names to their corresponding eigenvalues.
   (c) [8] Substitute the corresponding eigenvalues into the Hamiltonian, and argue that the remaining Hamiltonian is one whose eigenvalues and eigenstates you can find. You do not actually have to find these eigenvalues and eigenstates.

3. N identical spin-½ non-interacting particles lie in the ground state of a one-dimensional infinite square well of width L. For a single particle, the eigenstates are $\left|n,\pm\tfrac{1}{2}\right\rangle$, with energies $E_n = \pi^2 n^2\hbar^2 / \left(2mL^2\right)$.
   (a) [6] Would these particles be bosons or fermions? Which states would be occupied? You may assume N is even.
   (b) [10] What is the total energy for these particles? In the limit of large N, show your answer can be written in the form $E = \alpha N E_F$, where E_F is the energy of the highest occupied state, and α is a simple constant.
   (c) [9] Find the degeneracy "pressure", $P = -\partial E/\partial L$. Write your answer in terms of $\rho = N/L$ in the limit of large N.

Possibly helpful formulas: $\sum_{n=1}^{m} 1 = m$, $\sum_{n=1}^{m} n = \tfrac{1}{2}m(m+1)$, $\sum_{n=1}^{m} n^2 = \tfrac{1}{6}m(m+1)(2m+1)$.

4. Two particles are in a one-dimensional infinite square well with allowed region $-a < x < a$. In this region, the wave function is, $\psi(x_1, x_2) = N(x_1 \pm x_2)$ where N is a normalization constant. The $\pm$ simply means you are doing two problems at once, so it might be + or it might be − (You have to do both cases).
   (a) [7] What is the correct normalization constant N?
   (b) [7] What is the probability that particle one has positive position, $x_1 > 0$?
   (c) [8] What is the probability that both particles have positive position $x_1, x_2 > 0$?
   (d) [3] Suppose the particles are identical particles. What would be the appropriate sign for the $\pm$ if they are both bosons? If they are both fermions? Assume that any spin state would be symmetric, so their spin state looks like $|\chi, \chi\rangle$.

5. A harmonic oscillator is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|7\rangle + e^{i\theta}|8\rangle\right)$; unfortunately, the angle θ is uncertain, and is uniformly distributed across the interval $0 < \theta < \pi$.
   (a) [10] Find the state operator ρ. Your answer should look something like $\rho = \sum_{n,m} a_{n,m} |n\rangle\langle m|$. Check that $\mathrm{Tr}(\rho) = \sum_n \langle n|\rho|n\rangle = 1$.
   (b) [7] Find the expectation value of the Hamiltonian H for this state operator.
   (c) [8] Find the expectation value of the momentum P for this state operator.

6. The Kernel or propagator for the Harmonic oscillator with potential $\frac{1}{2}m\omega^2 x^2$ is given by

$$K(x,t;x_0,t_0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\left[\omega(t-t_0)\right]}} \exp\left[\frac{im\omega}{2\hbar \sin\left[\omega(t-t_0)\right]}\left\{\cos\left[\omega(t-t_0)\right]\left(x^2 + x_0^2\right) - 2xx_0\right\}\right].$$

   (a) [7] From this, deduce the free propagator in the limit $\omega \to 0$. Simplify as much as possible.
   (b) [18] At $t = 0$, the wave function is given by $\psi(x, t=0) = (A/\pi)^{1/4} e^{-Ax^2/2}$. Find the wave function at $\omega t = \frac{1}{2}\pi$. Simplify as much as possible.

Possibly Useful Formulas

1-D Harmonic Osc.

$V = \frac{1}{2}m\omega^2 x^2$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$

$P = i\sqrt{\frac{1}{2}\hbar m\omega}\left(a^\dagger - a\right)$

1-D infinite square well, size L

$E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2}$

State Operator

$\rho = \sum_i f_i |\psi_i\rangle\langle\psi_i|$

$\langle A\rangle = \mathrm{Tr}(\rho A)$

Clebsch-Gordan coeff. $\langle j_1, j_2; m_1, m_2 | j, m\rangle$

$\langle 1,\frac{1}{2};0,\frac{1}{2}|\frac{3}{2},\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}$, $\langle 1,\frac{1}{2};0,\frac{1}{2}|\frac{1}{2},\frac{1}{2}\rangle = -\sqrt{\frac{1}{3}}$,

$\langle 1,\frac{1}{2};1,-\frac{1}{2}|\frac{3}{2},\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}$, $\langle 1,\frac{1}{2};1,-\frac{1}{2}|\frac{1}{2},\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}$.

EM Fields

$\mathbf{B} = \nabla\times\mathbf{A}$

$\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla U$

EM Hamiltonian

$H = \frac{\boldsymbol{\pi}^2}{2m} - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S}$

Propagator:

$\Psi(x,t) = \int dx_0 K(x,t;x_0,t_0)\Psi(x_0,t_0)$

Possibly Useful Integral: $\int_{-\infty}^{\infty} e^{-\alpha y^2/2+\beta y} dy = \sqrt{2\pi/\alpha}\, e^{\beta^2/2\alpha}$