

Solutions to Final Exam, Fall 2022

Each question is worth 25 points, with points for each part marked separately. Some possibly useful formulas can be found at the end of the exam.

1. **A new particle has just been discovered! It is exactly like an electron *except* that in addition to having a Coulomb attraction to the proton $V_c(r) = -k_e e^2/r$, there is an additional repulsive term $V_r(r) = +6\hbar^2/\mu r^2$.**

- (a) [4] **Given that the potential depends only on r , what can we say about the bound states in terms of their dependence on angle and radius? Be as explicit as possible about the dependence on the angular part.**

Since it is spherically symmetric, the wave functions will automatically take the form

$$\Psi(r, \theta, \phi) = R(r) Y_l^m(\theta, \phi).$$

- (b) [5] **Write an ordinary differential equation for the radial wave function $R(r)$.**

The differential equation for R is given in the equations, it is

$$ER = -\frac{\hbar^2}{2\mu} \frac{1}{r} \frac{d^2}{dr^2}(rR) + \frac{(l^2 + l)\hbar^2}{2\mu r^2} R + VR = -\frac{\hbar^2}{2\mu} \frac{1}{r} \frac{d^2}{dr^2}(rR) + \frac{(l^2 + l)\hbar^2}{2\mu r^2} R + \frac{6\hbar^2}{\mu r^2} R - \frac{k_e e^2}{r} R,$$

$$ER = -\frac{\hbar^2}{2\mu} \frac{1}{r} \frac{d^2}{dr^2}(rR) + \frac{(l^2 + l + 12)\hbar^2}{2\mu r^2} R - \frac{k_e e^2}{r} R.$$

We note that the equation is identical to hydrogen, except that $l^2 + l$ has been modified to $l^2 + l + 12$.

- (c) [7] **Assume the ground state of modified hydrogen has $l = 0$. Based on this, show that the radial wave function for the *ground* state of modified hydrogen will be *exactly* like the equation for regular hydrogen with a different l value. What would be the corresponding l value for hydrogen?**

For the ground state, we expect $l = 0$ (which also implies $m = 0$). The equation for modified hydrogen for $l = 0$ states will therefore be

$$ER = -\frac{\hbar^2}{2\mu} \frac{1}{r} \frac{d^2}{dr^2}(rR) + \frac{12\hbar^2}{2\mu r^2} R - \frac{k_e e^2}{r} R.$$

This is identical to the equation for hydrogen with $l = 3$. Therefore, the solutions of the equation will be identical to the hydrogen wave functions with $l = 3$.

- (d) [5] Write the full wave function $\psi(r, \theta, \phi)$ for the ground state of modified hydrogen. Some hydrogen wave functions can be found on the equation sheet.

The lowest energy state for ordinary hydrogen for a given l will have the smallest n . But we always have $n > l$, so for $l = 3$ this would be $n = 4$. The lowest energy state will therefore be radial wave function $R_{43}(r)$. However, the angular wave function will have $l = 0, m = 0$, so we have

$$\psi(r, \theta, \phi) = R_{43}(r)Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \frac{r^3 e^{-r/4a}}{768\sqrt{35}a^9} = \frac{r^3 e^{-r/4a}}{1536\sqrt{35}\pi a^9}.$$

- (e) [4] For ordinary hydrogen, the energy is given by $E_n = -\frac{\alpha^2 \mu c^2}{2n^2}$. What is the energy of the ground state for modified hydrogen?

The ground state wave function satisfies the radial equation for $n = 4$, so even though it is the ground state, the energy will be modified to be $E_g = -\frac{1}{32}\alpha^2 \mu c^2$.

2. A single electron lies in a hydrogen atom in the state

$|\psi\rangle = \frac{1}{\sqrt{2}}|2, 0, 0, -\frac{1}{2}\rangle + \frac{i}{\sqrt{2}}|2, 1, +1, +\frac{1}{2}\rangle$, in standard notation $|n, l, m, m_s\rangle$, where n corresponds to the energy, l to L^2 , m to L_z and m_s to S_z . For each of the following operators, find the expectation values:

- (a) [5] $\langle L_z \rangle$ (b) [5] $\langle L^2 \rangle$ (c) [5] $\langle J_z \rangle$ (d) [10] $\langle J^2 \rangle$

The first term has eigenvalue $L_z = m\hbar = 0$ and the second $L_z = m\hbar = \hbar$, so we have

$$\begin{aligned} \langle L_z \rangle &= \langle \psi | L_z | \psi \rangle = \frac{1}{2} \left(\langle 2, 0, 0, -\frac{1}{2} | -i \langle 2, 1, +1, +\frac{1}{2} | \right) L_z \left(| 2, 0, 0, -\frac{1}{2} \rangle + i | 2, 1, +1, +\frac{1}{2} \rangle \right) \\ &= \frac{1}{2} \left(\langle 2, 0, 0, -\frac{1}{2} | -i \langle 2, 1, +1, +\frac{1}{2} | \right) \left(0 + i\hbar | 2, 1, +1, +\frac{1}{2} \rangle \right) = \frac{1}{2} \hbar. \end{aligned}$$

For L^2 , the first term has eigenvalue $L^2 = \hbar^2(l^2 + l) = 0$ and the second

$L^2 = \hbar^2(l^2 + l) = 2\hbar^2$, so we have

$$\begin{aligned} \langle L^2 \rangle &= \langle \psi | L^2 | \psi \rangle = \frac{1}{2} \left(\langle 2, 0, 0, -\frac{1}{2} | -i \langle 2, 1, +1, +\frac{1}{2} | \right) L^2 \left(| 2, 0, 0, -\frac{1}{2} \rangle + i | 2, 1, +1, +\frac{1}{2} \rangle \right) \\ &= \frac{1}{2} \left(\langle 2, 0, 0, -\frac{1}{2} | -i \langle 2, 1, +1, +\frac{1}{2} | \right) \left(0 + i2\hbar^2 | 2, 1, +1, +\frac{1}{2} \rangle \right) = \frac{1}{2} \cdot 2\hbar^2 = \hbar^2. \end{aligned}$$

Calculating $\langle J_z \rangle$ is slightly trickier. We first note that $J_z = L_z + S_z$, and therefore its eigenvalues are $J_z = \hbar m + \hbar m_s$, so the first term has eigenvalue $J_z = \hbar(0 - \frac{1}{2}) = -\frac{1}{2}\hbar$, and the second term has eigenvalue $J_z = \hbar(1 + \frac{1}{2}) = \frac{3}{2}\hbar$. We therefore have

$$\begin{aligned}\langle J_z \rangle &= \langle \psi | J_z | \psi \rangle = \frac{1}{2} \left(\langle 2, 0, 0, -\frac{1}{2} | -i \langle 2, 1, +1, +\frac{1}{2} | \right) J_z \left(| 2, 0, 0, -\frac{1}{2} \rangle + i | 2, 1, +1, +\frac{1}{2} \rangle \right) \\ &= \frac{1}{2} \left(\langle 2, 0, 0, -\frac{1}{2} | -i \langle 2, 1, +1, +\frac{1}{2} | \right) \left(-\frac{1}{2} \hbar | 2, 0, 0, -\frac{1}{2} \rangle + i \frac{3}{2} \hbar | 2, 1, +1, +\frac{1}{2} \rangle \right) = \frac{1}{2} \cdot \left(-\frac{1}{2} \hbar + \frac{3}{2} \hbar \right) = \frac{1}{2} \hbar.\end{aligned}$$

Calculating $\langle \mathbf{J}^2 \rangle$ seems much harder. We need to know the quantum number j for each state, but they aren't even written as eigenstates of \mathbf{J}^2 , which makes this tricky, and makes it sound like we are going to need Clebsch-Gordan coefficients. For the first term, we are combining an $l = 0$ state with an electron with $s = \frac{1}{2}$, and these combine to make states with $j = \frac{1}{2}$ as the only possibility, so this state is an eigenstate of \mathbf{J}^2 . For the second term, we are combining an $l = 1$ state with an electron with $s = \frac{1}{2}$, so this time we could have $j = \frac{1}{2}$ or $j = \frac{3}{2}$. However, because we now this state has $J_z = \hbar(1 + \frac{1}{2}) = \frac{3}{2} \hbar$, $m_j = \frac{3}{2}$, but $j \geq |m_j| = \frac{3}{2}$. So the only option is $j = \frac{3}{2}$. Hence the states have eigenvalues $\mathbf{J}^2 = \hbar^2(j^2 + j) = \frac{3}{4} \hbar^2$ for the first state and $\mathbf{J}^2 = \hbar^2(j^2 + j) = \frac{15}{4} \hbar^2$ for the second. Hence we have

$$\begin{aligned}\langle \mathbf{J}^2 \rangle &= \langle \psi | \mathbf{J}^2 | \psi \rangle = \frac{1}{2} \left(\langle 2, 0, 0, -\frac{1}{2} | -i \langle 2, 1, +1, +\frac{1}{2} | \right) \mathbf{J}^2 \left(| 2, 0, 0, -\frac{1}{2} \rangle + i | 2, 1, +1, +\frac{1}{2} \rangle \right) \\ &= \frac{1}{2} \left(\langle 2, 0, 0, -\frac{1}{2} | -i \langle 2, 1, +1, +\frac{1}{2} | \right) \left(\frac{3}{4} \hbar^2 | 2, 0, 0, -\frac{1}{2} \rangle + i \frac{15}{4} \hbar^2 | 2, 1, +1, +\frac{1}{2} \rangle \right) = \frac{1}{2} \cdot \left(\frac{3}{4} \hbar^2 + \frac{15}{4} \hbar^2 \right) \\ &= \frac{9}{4} \hbar^2.\end{aligned}$$

3. An electron of mass m lies in a region with electric field $\mathbf{E} = m\omega_0^2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})/e$ and magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

(a) [7] Show that these can be obtained with a scalar potential $U = a(x^2 + y^2)$ and vector potential $\mathbf{A}(\mathbf{r}) = b(x\hat{\mathbf{y}} - y\hat{\mathbf{x}})$, and find the coefficients a and b .

The electric field is given by $\mathbf{E} = -\nabla U - \frac{\partial \mathbf{A}}{\partial t} = -a\nabla(x^2 + y^2) = -2a(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})$.

Comparison with the desired form tells us $a = -m\omega_0^2/2e$. The magnetic field is given by $\mathbf{B} = \nabla \times \mathbf{A}$, whose components work out to

$$B_x = \frac{\partial}{\partial y} 0 - \frac{\partial}{\partial z}(bx) = 0, \quad B_y = \frac{\partial}{\partial z}(-by) - \frac{\partial}{\partial z} 0 = 0, \quad B_z = \frac{\partial}{\partial x}(bx) - \frac{\partial}{\partial y}(-bx) = 2b.$$

Matching with the desired form, we see that $b = B/2$. So our scalar and vector potentials are

$$U = -\frac{m\omega_0^2}{2e}(x^2 + y^2), \quad \mathbf{A} = \frac{B}{2}(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}).$$

(b) [7] Write the Hamiltonian explicitly. It may be helpful to expand it out.

The Hamiltonian is given in the equations, but we want to substitute $\boldsymbol{\pi} = \mathbf{P} + e\mathbf{A}$ in explicitly. We find

$$\begin{aligned} H &= \frac{\boldsymbol{\pi}^2}{2m} - eU + \frac{ge}{2m} \mathbf{B} \cdot \mathbf{S} = \frac{1}{2m} \left[(P_x - \frac{1}{2}eBY)^2 + (P_y + \frac{1}{2}eBX)^2 + P_z^2 \right] + \frac{1}{2}m\omega_0^2(X^2 + Y^2) + \frac{ge}{2m}BS_z \\ &= \frac{1}{2m} \left[\mathbf{P}^2 + eB(XP_y - YP_x) + \frac{1}{4}e^2B^2(X^2 + Y^2) \right] + \frac{1}{2}m\omega_0^2(X^2 + Y^2) + \frac{ge}{2m}BS_z \\ &= \frac{\mathbf{P}^2}{2m} + \frac{eB}{2m}L_z + \left(\frac{e^2B^2}{8m} + \frac{1}{2}m\omega_0^2 \right) (X^2 + Y^2) + \frac{ge}{2m}BS_z \end{aligned}$$

(c) [6] Demonstrate that this Hamiltonian commutes with one of the three momentum operators \mathbf{P} , one of the three angular momentum operators \mathbf{L} , and one of the three spin operators \mathbf{S} . Which one, in each case? Do they commute with each other?

This first term commutes with all components of \mathbf{P} , as does the last. The third term is has no z 's in it, so it commutes with P_z , and L_z also does. So P_z commutes.

All the angular momentum operators \mathbf{L} commute with the first and last term, and because the third term is rotationally invariant about the z -axis, it commutes with L_z . Obviously, so does the second term, so L_z commutes.

All components of the spin \mathbf{S} commute with the first three terms, but S_z is the only one that commuted with the last term.

In summary, the three operators $\{P_z, L_z, S_z\}$ all commute with the Hamiltonian. They also commute with each other, so they can be simultaneously diagonalized.

(d) [5] Call the three eigenvalues under the operators $\hbar k$, $\hbar m$, and $\hbar m_s$ respectively. Are there any restrictions on these eigenvalues?

If we call the eigenvalues of $\{P_z, L_z, S_z\}$ the indicated values, then we can find solutions such that $P_z|\psi\rangle = \hbar k|\psi\rangle$, $L_z|\psi\rangle = \hbar m|\psi\rangle$ and $S_z|\psi\rangle = \hbar m_s|\psi\rangle$. Angular momentum around an axis *always* comes in integer values, so m is an integer, and the z -component of spin has eigenvalues correspond to $m_s = \pm \frac{1}{2}$. However, there is no restriction on k .

4. A free particle has wave function at $t = 0$ given by $\psi(x, t = 0) = Ne^{-Ax^2/2}$.

(a) [14] Find the wave function at arbitrary time t . You may wish to simplify a bit.

We are given the initial wave function $\psi(x_0, t_0) = Ne^{-Ax_0^2/2}$ where $t_0 = 0$. We substitute it into the general formula and then use the given integral to help us do the integral. The key is to split things up in the exponent into terms with different powers of x_0 . We have

$$\begin{aligned}\Psi(x,t) &= \int dx_0 K(x,t;x_0,t_0) \Psi(x_0,t_0) = N \sqrt{\frac{m}{2\pi i \hbar t}} \int dx_0 \exp\left[\frac{im(x-x_0)^2}{2\hbar t} - \frac{1}{2} Ax_0^2\right] \\ &= N \sqrt{\frac{m}{2\pi i \hbar t}} \int dx_0 \exp\left[\left(\frac{im}{2\hbar t} - \frac{A}{2}\right)x_0^2 - \frac{imx}{\hbar t}x_0 + \frac{imx^2}{2\hbar t}\right].\end{aligned}$$

This is now exactly the form of the integral that is given, with $\alpha = A - im/\hbar t$ and $\beta = imx/\hbar t$, and the last term in the exponential simply going along for the ride. We therefore have

$$\begin{aligned}\Psi(x,t) &= N \sqrt{\frac{m}{2\pi i \hbar t}} \sqrt{\frac{2\pi}{A - im/\hbar t}} \exp\left[\frac{(imx/\hbar t)^2}{2(A - im/\hbar t)} + \frac{imx^2}{2\hbar t}\right] \\ &= N \sqrt{\frac{m}{Ai\hbar t + m}} \exp\left[-\frac{m^2 x^2}{2\hbar t(A\hbar t - im)} + \frac{imx^2}{2\hbar t}\right] \\ &= N \sqrt{\frac{m}{m + Ai\hbar t}} \exp\left[\frac{-m^2 x^2 + imx^2(A\hbar t - im)}{2\hbar t(A\hbar t - im)}\right] = N \sqrt{\frac{m}{m + Ai\hbar t}} \exp\left[\frac{imx^2 A\hbar t}{2\hbar t(A\hbar t - im)}\right] \\ &= N \sqrt{\frac{m}{m + Ai\hbar t}} \exp\left[\frac{-Amx^2}{2(m + iA\hbar t)}\right].\end{aligned}$$

(b) [5] Show that at $t = 0$, the formula reduces to wave function given at $t = 0$.

Because we did a lot of simplification, we can see that at $t = 0$, the formula simplifies to

$$\Psi(x, t = 0) = N \sqrt{\frac{m}{m}} \exp\left[\frac{-Amx^2}{2m}\right] = N e^{-Ax^2/2}.$$

(c) [6] Find the probability density as a function of x at all times t . For full credit, simplify your answer as much as possible and make sure the result is real.

The probability density is $\Psi\Psi^*$, which is

$$\begin{aligned}|\Psi|^2 &= NN^* \sqrt{\frac{m}{m + Ai\hbar t}} \sqrt{\frac{m}{m - Ai\hbar t}} \exp\left[-\frac{Amx^2}{2(m + iA\hbar t)} - \frac{Amx^2}{2(m - iA\hbar t)}\right] \\ &= \frac{m|N|^2}{\sqrt{m^2 + A^2\hbar^2 t^2}} \exp\left[-\frac{Amx^2(m - iA\hbar t + m + iA\hbar t)}{2(m^2 + A^2\hbar^2 t^2)}\right] = \frac{m|N|^2}{\sqrt{m^2 + A^2\hbar^2 t^2}} \exp\left(-\frac{Am^2 x^2}{m^2 + A^2\hbar^2 t^2}\right).\end{aligned}$$

This is manifestly real.

5. Twelve (12) non-interacting identical particles with spin are all in a 2D harmonic oscillator with angular frequency ω , so the states would be labeled something like $|n_x, n_y, \chi\rangle$.

(a) [4] What is the energy of a single particle in an arbitrary state of this type?

This is pretty trivial; the energy of a single particle for the harmonic oscillator states will just be $E = \hbar\omega\left(n_x + \frac{1}{2}\right) + \hbar\omega\left(n_y + \frac{1}{2}\right) = \hbar\omega\left(n_x + n_y + 1\right)$. It is independent of spin.

(b) [5] If the particles have spin 0, would they be bosons or fermions? Which state(s) would be occupied in the ground state? What would be the ground state energy?

Spin 0 particles are bosons, which do not need to satisfy the Pauli exclusion principle. Hence they will all pile into the ground state $n_x = n_y = 0$. Since this particle has no spin, you don't need to specify the spin state, so the only occupied states are $\{|0, 0\rangle\}$. These each have an energy of $\hbar\omega$, so the total energy is $E = 12\hbar\omega$.

(c) [8] If the particles have spin $\frac{1}{2}$, would they be bosons or fermions? Which state(s) would be occupied in the ground state? What would be the ground state energy?

These are fermions, and because they are spin $\frac{1}{2}$, they can take on the values $m_s = \pm\frac{1}{2}$. We normally abbreviate this as simply \pm . Noting that we get two states for each pair n_x, n_y , we realize that we need to fill up six n_x, n_y pairs, choosing them so their sum is as small as possible. The lowest energy states will be $\{|0, 0, \pm\rangle, |1, 0, \pm\rangle, |0, 1, \pm\rangle, |2, 0, \pm\rangle, |1, 1, \pm\rangle, |0, 2, \pm\rangle\}$. The first two particles will have energy $\hbar\omega$ each, the next four will have energy $2\hbar\omega$, and the last six will have energy $3\hbar\omega$, so the total energy will be

$$E = 2(\hbar\omega) + 4(2\hbar\omega) + 6(3\hbar\omega) = 28\hbar\omega.$$

(d) [8] If the particles have spin $\frac{3}{2}$, would they be bosons or fermions? Which state(s) would be occupied in the ground state? What would be the ground state energy?

Again these are fermions, and because they are spin $\frac{3}{2}$, they can take on the values $m_s = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$. Noting that we get four states for each pair n_x, n_y , we realize that we need to fill up three n_x, n_y pairs, choosing them so their sum is as small as possible. The lowest energy states will be $\{|0, 0, m_s\rangle, |1, 0, m_s\rangle, |0, 1, m_s\rangle\}$, where m_s takes on the four indicated values. The first four particles have energy $\hbar\omega$ and the other eight have energy $2\hbar\omega$, so the total energy is

$$E = 4(\hbar\omega) + 8(2\hbar\omega) = 20\hbar\omega.$$

6. A spin $\frac{1}{2}$ -particle is in one of two states, given by $\Psi_{\pm} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2i \pm 1 \end{pmatrix}$, with equal probability.

(a) [8] Find the state operator ρ . As a check, find its trace.

The state operator is given by

$$\begin{aligned} \rho &= \sum_i f_i |\psi_i\rangle \langle \psi_i| = \frac{1}{2} \cdot \frac{1}{6} \left[\begin{pmatrix} 1 \\ 2i+1 \end{pmatrix} (1 \quad -2i+1) + \begin{pmatrix} 1 \\ 2i-1 \end{pmatrix} (1 \quad -2i-1) \right] \\ &= \frac{1}{12} \left[\begin{pmatrix} 1 & -2i+1 \\ 2i+1 & 5 \end{pmatrix} + \begin{pmatrix} 1 & -2i-1 \\ 2i-1 & 5 \end{pmatrix} \right] = \frac{1}{12} \begin{pmatrix} 2 & -4i \\ 4i & 10 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & -2i \\ 2i & 5 \end{pmatrix}. \end{aligned}$$

The trace is $\text{Tr}(\rho) = \frac{1}{6} + \frac{5}{6} = 1$, as it must.

(b) [9] Find the expectation value of all three spin operators $S_i = \frac{1}{2} \hbar \sigma_i$, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We use the formula $\langle A \rangle = \text{Tr}(\rho A)$, which gives us

$$\langle S_x \rangle = \text{Tr}(\rho \frac{1}{2} \hbar \sigma_x) = \frac{1}{2} \cdot \frac{1}{6} \hbar \text{Tr} \left[\begin{pmatrix} 1 & -2i \\ 2i & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \frac{1}{12} \hbar \text{Tr} \begin{pmatrix} -2i & 1 \\ 5 & 2i \end{pmatrix} = 0,$$

$$\langle S_y \rangle = \text{Tr}(\rho \frac{1}{2} \hbar \sigma_y) = \frac{1}{2} \cdot \frac{1}{6} \hbar \text{Tr} \left[\begin{pmatrix} 1 & -2i \\ 2i & 5 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = \frac{1}{12} \hbar \text{Tr} \begin{pmatrix} 2 & -i \\ 5i & 2 \end{pmatrix} = \frac{4}{12} \hbar = \frac{1}{3} \hbar,$$

$$\langle S_z \rangle = \text{Tr}(\rho \frac{1}{2} \hbar \sigma_z) = \frac{1}{2} \cdot \frac{1}{6} \hbar \text{Tr} \left[\begin{pmatrix} 1 & -2i \\ 2i & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \frac{1}{12} \hbar \text{Tr} \begin{pmatrix} 1 & 2i \\ 2i & -5 \end{pmatrix} = -\frac{4}{12} \hbar = -\frac{1}{3} \hbar.$$

(c) [8] The Hamiltonian is given by $H = \omega(S_z - S_y)$. Prove or disprove that $d\rho/dt = 0$.

We use the formula given,

$$\begin{aligned} \frac{d}{dt} \rho &= \frac{1}{i\hbar} [H, \rho] = \frac{\hbar\omega}{i\hbar} [\sigma_z - \sigma_y, \rho] = \frac{\omega}{12i} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & -2i \\ 2i & 5 \end{pmatrix} \right] \\ &= \frac{\omega}{12i} \left\{ \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix} \begin{pmatrix} 1 & -2i \\ 2i & 5 \end{pmatrix} - \begin{pmatrix} 1 & -2i \\ 2i & 5 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix} \right\} = \frac{\omega}{12i} \left\{ \begin{pmatrix} -1 & 3i \\ -3i & -7 \end{pmatrix} - \begin{pmatrix} -1 & 3i \\ -3i & -7 \end{pmatrix} \right\} = 0. \end{aligned}$$

Since the time derivative is zero, it doesn't change.

Possibly Useful Formulas

<p style="text-align: center;">Free Propagator:</p> $\Psi(x, t) = \int dx_0 K(x, t; x_0, t_0) \Psi(x_0, t_0)$ $K(x, t; x_0, t_0) = \sqrt{\frac{m}{2\pi i \hbar (t - t_0)}} \exp\left[\frac{im(x - x_0)^2}{2\hbar(t - t_0)}\right]$		<p style="text-align: center;">Hydrogen Wave Functions</p> $R_{10} = \frac{2e^{-r/a}}{a^{3/2}}, \quad R_{30} = \frac{2e^{-r/3a}}{3\sqrt{3}a^3} \left(1 - \frac{2r}{3a} + \frac{2r^2}{27a^2}\right)$ $R_{32} = \frac{2\sqrt{2}r^2 e^{-r/3a}}{81\sqrt{15}a^7}, \quad R_{43} = \frac{r^3 e^{-r/4a}}{768\sqrt{35}a^9}$ $R_{40} = \frac{e^{-r/4a}}{4\sqrt{a^3}} \left(1 - \frac{3r}{4a} + \frac{r^2}{8a^2} - \frac{r^3}{192a^3}\right)$	
<p style="text-align: center;">Radial Wave Equation</p> $ER = -\frac{\hbar^2}{2\mu} \frac{1}{r} \frac{d^2}{dr^2}(rR) + \frac{(l^2 + l)\hbar^2}{2\mu r^2} R + VR$		<p style="text-align: center;">Spherical Harmonics</p> $Y_0^0 = \frac{1}{\sqrt{4\pi}}$ $Y_2^2 = \sqrt{\frac{15}{32\pi}} e^{2i\phi} \sin^2 \theta$ $Y_3^3 = \frac{\sqrt{35}}{8\sqrt{\pi}} e^{3i\phi} \sin^3 \theta$	
<p style="text-align: center;">EM Fields</p> $\mathbf{B} = \nabla \times \mathbf{A}$ $\mathbf{E} = -\partial \mathbf{A} / \partial t - \nabla U$	<p style="text-align: center;">EM Hamiltonian</p> $H = \frac{\pi^2}{2m} - eU + \frac{ge}{2m} \mathbf{B} \cdot \mathbf{S}$	<p style="text-align: center;">State Operator</p> $\rho = \sum_i f_i \psi_i\rangle \langle \psi_i $ $\langle A \rangle = \text{Tr}(\rho A)$ $i\hbar \frac{d}{dt} \rho = [H, \rho]$	

Possibly Useful Integral:

$$\int_{-\infty}^{\infty} e^{-\alpha y^2 / 2 - \beta y} dy = \sqrt{2\pi/\alpha} e^{\beta^2 / 2\alpha}$$