Physics 741 – Graduate Quantum Mechanics 1

Solutions to Final Exam, Fall 2018

Each question is worth 25 points, with points for each part marked separately. Some possibly useful formulas can be found at the end of the exam.

1. A hydrogen atom is in the state $\left|\psi\right\rangle=\sqrt{\frac{1}{3}}\left|2,1,0,\frac{1}{2}\right\rangle+\sqrt{\frac{2}{3}}\left|2,1,1,-\frac{1}{2}\right\rangle$, where we are using the notation $\left|n,l,m,m_s\right\rangle$, where *l*, *m*, and *m* correspond to $\mathbf{L}^2$, L_z, and S_z respectively.

(a) [5] If one were to measure the operators L_z, and S_z, what would be the possible outcomes and corresponding probabilities?

The states given $\left|n,l,m,m_s\right\rangle$ are eigenstates of L_z, and S_z with eigenvalues $\hbar m$ and $\hbar m_s$ respectively. The state $\left|2,1,0,\frac{1}{2}\right\rangle$ has $L_z=0$ and $S_z=\frac{1}{2}\hbar$, while $\left|2,1,1,-\frac{1}{2}\right\rangle$ has $L_z=\hbar$ and $S_z=-\frac{1}{2}\hbar$. The corresponding probabilities are

$$P\left(L_z=0\right)=P\left(S_z=\tfrac{1}{2}\hbar\right)=\left|\left\langle 2,1,0,\tfrac{1}{2}\middle|\psi\right\rangle\right|^2=\left(\sqrt{\tfrac{1}{3}}\right)^2=\tfrac{1}{3},$$

$$P\left(L_z=\hbar\right)=P\left(S_z=-\tfrac{1}{2}\hbar\right)=\left|\left\langle 2,1,1,-\tfrac{1}{2}\middle|\psi\right\rangle\right|^2=\left(\sqrt{\tfrac{2}{3}}\right)^2=\tfrac{2}{3}.$$

(b)[20] If one were to measure $\mathbf{J}^2$ and J_z, what would be the possible outcomes and corresponding probabilities?

The two states $\left|2,1,0,\frac{1}{2}\right\rangle$ and $\left|2,1,1,-\frac{1}{2}\right\rangle$ each have $J_z=\hbar\left(m+m_s\right)=\frac{1}{2}\hbar$; hence the outcome of a measurement of J_z is automatically $\frac{1}{2}\hbar$, so $P\left(J_z=\frac{1}{2}\hbar\right)=1$.

To find the probability that the outcome of a measurement of $\mathbf{J}^2$ has a particular value $\hbar^2\left(j^2+j\right)$, we must find the overlap $\left\langle n,l,j,m_j\middle|\psi\right\rangle$, sum it over values of n, l, and m_j, and then square its amplitude. However, in this case, we know that $n = 2$, $l = 1$, and $m_j=\frac{1}{2}$, so there is only a single term. Knowing that we are adding orbital angular momentum 1 to spin ½, the only outcomes are $j=\left|1-\frac{1}{2}\right|,\ldots,1+\frac{1}{2}=\frac{1}{2},\frac{3}{2}$, the only possible outcomes are $\mathbf{J}^2=\frac{3}{4}\hbar^2$ and $\mathbf{J}^2=\frac{15}{4}\hbar^2$. The corresponding amplitudes then work out to just be Clebsch-Gordan coefficients, so $\left\langle n,l,j,m_j\middle|n,l,m,m_s\right\rangle=\left\langle 1,\frac{1}{2};m,m_s\middle|j,m_j\right\rangle$. So we have

$$P\left(J_z=\tfrac{3}{4}\hbar^2\right)=\left|\left\langle 2,1,\tfrac{1}{2},\tfrac{1}{2}\middle|\psi\right\rangle\right|^2=\left|\sqrt{\tfrac{1}{3}}\left\langle 2,1,\tfrac{1}{2},\tfrac{1}{2}\middle|2,1,0,\tfrac{1}{2}\right\rangle+\sqrt{\tfrac{2}{3}}\left\langle 2,1,\tfrac{1}{2},\tfrac{1}{2}\middle|2,1,1,-\tfrac{1}{2}\right\rangle\right|^2$$
$$=\left|\sqrt{\tfrac{1}{3}}\left\langle 1,\tfrac{1}{2};0,\tfrac{1}{2}\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle+\sqrt{\tfrac{2}{3}}\left\langle 1,\tfrac{1}{2};1,-\tfrac{1}{2}\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right|^2=\left|-\sqrt{\tfrac{1}{3}}\sqrt{\tfrac{1}{3}}+\sqrt{\tfrac{2}{3}}\sqrt{\tfrac{2}{3}}\right|^2=\left|\tfrac{1}{3}\right|^2=\tfrac{1}{9},$$

$$P\left(J_z=\tfrac{15}{4}\hbar^2\right)=\left|\left\langle 2,1,\tfrac{3}{2},\tfrac{1}{2}\middle|\psi\right\rangle\right|^2=\left|\sqrt{\tfrac{1}{3}}\left\langle 2,1,\tfrac{3}{2},\tfrac{1}{2}\middle|2,1,0,\tfrac{1}{2}\right\rangle+\sqrt{\tfrac{2}{3}}\left\langle 2,1,\tfrac{3}{2},\tfrac{1}{2}\middle|2,1,1,-\tfrac{1}{2}\right\rangle\right|^2$$
$$=\left|\sqrt{\tfrac{1}{3}}\left\langle 1,\tfrac{1}{2};0,\tfrac{1}{2}\middle|\tfrac{3}{2},\tfrac{1}{2}\right\rangle+\sqrt{\tfrac{2}{3}}\left\langle 1,\tfrac{1}{2};1,-\tfrac{1}{2}\middle|\tfrac{3}{2},\tfrac{1}{2}\right\rangle\right|^2=\left|\sqrt{\tfrac{2}{3}}\sqrt{\tfrac{1}{3}}+\sqrt{\tfrac{1}{3}}\sqrt{\tfrac{2}{3}}\right|^2=\left|\tfrac{2\sqrt{2}}{3}\right|^2=\tfrac{8}{9}.$$

2. An electron is in a region with electric field $\mathbf{E} = E_0\hat{\mathbf{x}}$ and magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$.

(a) [8] Find an electrostatic potential U that alone can account for this electric field. Which coordinate(s) is it independent of? Find a vector potential A that is independent of the same coordinate(s) and accounts for the magnetic field.

Since there is no time-dependance, it seems reasonable to pick U and **A** so that they are both time-independent. We therefore want to have $\mathbf{E} = -\nabla U$, so we need $E_0 = -\partial U/\partial x$. The simplest solution is just $U = -E_0 x$. This is independent of both *y* and *z*, so it seems like it might be a good idea to pick **A** to depend only on *x*. To get **B** in the *z*-direction, we want **A** to exist in a perpendicular direction. Since one of the contributions to B_z is $\partial A_y/\partial x$, we therefore have $\partial A_y/\partial x = B_0$, and we guess $\mathbf{A} = B_0 x\hat{\mathbf{y}}$. We therefore have

$$\mathbf{A} = B_0 x\hat{\mathbf{y}} \quad \text{and} \quad U = -E_0 x.$$

(b) [9] Write the Hamiltonian explicitly. Find three operators that commute with the Hamiltonian and with each other. These operators might be spin operators, momentum operators, or angular momentum operators. Give names to their corresponding eigenvalues.

We use the formula given in the equations together with $\boldsymbol{\pi} = \mathbf{P} + e\mathbf{A}$ to get

$$H = \frac{\boldsymbol{\pi}^2}{2m} - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S} = \frac{1}{2m}\left(\mathbf{P} + eB_0X\hat{\mathbf{y}}\right)^2 + eE_0X + \frac{ge}{2m}B_0S_z$$
$$= \frac{1}{2m}\left[P_x^2 + \left(P_y + eB_0X\right)^2 + P_z^2\right] + eE_0X + \frac{ge}{2m}B_0S_z$$

We note that this Hamiltonian commutes with P_y, P_z, and S_z, and we label the corresponding eigenvalues as $\hbar k_y$, $\hbar k_z$, and $\hbar m_s$. There is no restriction on k_y or k_z, but m_s can only take on the values $\pm\frac{1}{2}$.

(c) [8] Substitute the corresponding eigenvalues into the Hamiltonian, and argue that the remaining Hamiltonian is one whose eigenvalues and eigenstates you can find. You do not actually have to find these eigenvalues and eigenstates.

Substituting in the corresponding eigenvalues, we have

$$H = \frac{1}{2m}\left[P_x^2 + \left(\hbar k_y + eB_0X\right)^2 + \hbar^2 k_z^2\right] + eE_0X + \frac{ge}{2m}B_0\hbar m_s$$

This can be written as the kinetic energy of a particle in one dimension together with a potential that, when squared out, is just a quadratic in *X*. Completing the square will turn this problem into a 1D-harmonic oscillator that is centered somewhere away from the origin. Of course, we know how to solve the harmonic oscillator, though in this case writing the states out would be a pain.

3. ***N*** **identical spin-½ non-interacting particles lie in the ground state of a one-dimensional infinite square well of width *L*. For a single particle, the eigenstates are $\left|n,\pm\frac{1}{2}\right\rangle$, with energies $E_n = \pi^2 n^2 \hbar^2 / \left(2mL^2\right)$.**

(a) [6] Would these particles be bosons or fermions? Which states would be occupied? You may assume *N* is even.

Since they have spin ½, they will be fermions, which satisfy the Pauli exclusion principle. The states what will be occupied are the first *N* state, which, since there are two states for each value of *n*, will be the states $\left|n,\pm\frac{1}{2}\right\rangle$ for $n = 1, 2, \ldots, \frac{1}{2}N$. If *N* were odd, the state $n = \frac{1}{2}(N+1)$ would be half-filled, with all lower states filled.

(b) [10] What is the total energy for these particles? In the limit of large *N*, show your answer can be written in the form $E = \alpha N E_F$, where E_F is the energy of the highest occupied state, and α is a simple constant.

The Fermi energy is just E_n for the highest state occupied, which is $n = \frac{1}{2}N$, so

$$E_F = \frac{\pi^2\hbar^2 (N/2)^2}{2mL^2} = \frac{\pi^2\hbar^2 N^2}{8mL^2}$$

The total energy is the sum of the energy of all the particles. We multiply by 2 to account for the two spin states, and add up the energies for each particle, to give us

$$E = 2\sum_{n=1}^{N/2} \frac{\pi^2\hbar^2 m^2}{2mL^2} = 2\frac{\pi^2\hbar^2}{2mL^2}\frac{1}{6}\left(\frac{N}{2}\right)\left(\frac{N}{2}+1\right)\left(2\frac{N}{2}+1\right) = \frac{\pi\hbar^2}{24mL^2}(N)(N+2)(N+1)$$

$$= \frac{\pi^2\hbar^2}{24mL^2}N(N+2)(N+1) = \frac{1}{3}NE_F\left(1+\frac{2}{N}\right)\left(1+\frac{1}{N}\right).$$

In the limit $N\to\infty$, this becomes $E = \frac{1}{3}NE_F$.

(c) [9] Find the degeneracy "pressure", $P = -\partial E/\partial L$. Write your answer in terms of $\rho = N/L$ in the limit of large *N*.

Taking the derivattive is trivial; we find

$$P = -\frac{\partial E}{\partial L} = \frac{\pi^2\hbar^2 N(N+1)(N+2)}{12mL^3} = \frac{\pi^2\hbar^2}{12m}\rho^3\left(1+\frac{1}{N}\right)\left(1+\frac{2}{N}\right)$$

In the limit $N\to\infty$, this becomes $P = \pi^2\hbar^2\rho^3/12m$.

Possibly helpful formulas: $\sum_{n=1}^{m} 1 = m$, $\sum_{n=1}^{m} n = \frac{1}{2}m(m+1)$, $\sum_{n=1}^{m} n^2 = \frac{1}{6}m(m+1)(2m+1)$.

4. Two particles are in a one-dimensional infinite square well with allowed region $-a < x < a$. In this region, the wave function is, $\psi(x_1, x_2) = N(x_1 \pm x_2)$ where N is a normalization constant. The $\pm$ simply means you are doing two problems at once, so it might be + or it might be $-$ (you have to do both cases).

(a) [7] What is the correct normalization constant N?

The nomalization condition is

$$1 = \int \left|\psi(x_1, x_2)\right|^2 dx_1 dx_2 = N^2 \int_{-a}^{a} dx_1 \int_{-a}^{a} dx_2 \left(x_1 \pm x_2\right)^2 = N^2 \int_{-a}^{a} dx_1 \int_{-a}^{a} dx_2 \left(x_1^2 \pm 2x_1x_2 + x_2^2\right)$$

$$= N^2 \int_{-a}^{a} dx_1 \left(x_1^2 x_2 \pm \tfrac{2}{2} x_1 x_2^2 + \tfrac{1}{3} x_2^3\right)\Big|_{x_2=-a}^{a} = N^2 \int_{-a}^{a} dx_1 \left(2ax_1^2 + \tfrac{2}{3}a^3\right) = N^2 \left(\tfrac{2}{3}ax_1^3 + \tfrac{2}{3}a^3 x_1\right)\Big|_{-a}^{a}$$

$$= N^2 \left(\tfrac{4}{3}a^4 + \tfrac{4}{3}a^4\right) = \tfrac{8}{3}N^2 a^4 .$$

So $N = \sqrt{3/8a^4}$

(b) [7] What is the probability that particle one has positive position, $x_1 > 0$?

The only thing that needs to change from the previous computation is that the limits on x_1 need to change, and we substitute the value of N computed in part (a). The probability is

$$P(x_1 > 0) = N^2 \int_{0}^{a} dx_1 \int_{-a}^{a} dx_2 \left(x_1 \pm x_2\right)^2 = \cdots = N^2 \left(\tfrac{2}{3}ax_1^3 + \tfrac{2}{3}a^3 x_1\right)\Big|_{0}^{a} = \frac{3}{8a^4}\left(\frac{2}{3}a^4 + \frac{2}{3}a^4\right) = \frac{1}{2}.$$

(c) [8] What is the probability that both particles have positive position $x_1, x_2 > 0$?

This time we need to change both integral limits, which means we will have to back up and do more work this time. We have

$$P(x_1, x_2 > 0) = N^2 \int_{0}^{a} dx_1 \int_{0}^{a} dx_2 \left(x_1 \pm x_2\right)^2 = N^2 \int_{0}^{a} dx_1 \left(x_1^2 x_2 \pm \tfrac{2}{2} x_1 x_2^2 + \tfrac{1}{3} x_2^3\right)\Big|_{x_2=0}^{a}$$

$$= N^2 \int_{0}^{a} dx_1 \left(ax_1^2 \pm a^2 x_1 + \tfrac{1}{3}a^3\right) = N^2 \left(\tfrac{1}{3}ax_1^3 \pm \tfrac{1}{2}a^2 x_1^2 + \tfrac{1}{3}a^3 x_1\right)\Big|_{0}^{a}$$

$$= N^2 \left(\tfrac{1}{3}a^4 \pm \tfrac{1}{2}a^4 + \tfrac{1}{3}a^4\right) = \frac{3a^4}{8a^4}\left(\frac{2}{3} \pm \frac{1}{2}\right) = \frac{1}{4} \pm \frac{3}{16} = \begin{cases} \tfrac{7}{16} & \text{for } +, \\ \tfrac{1}{16} & \text{for } -. \end{cases}$$

(d) [3] Suppose the particles are identical particles. What would be the appropriate sign for the $\pm$ if they are both bosons? If they are both fermions? Assume that any spin state would be symmetric, so their spin state looks like $\left|\chi, \chi\right\rangle$.

The plus sign is appropriate for bosons, and the minus sign for bosons.

5. A harmonic oscillator is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|7\rangle + e^{i\theta}|8\rangle\right)$; unfortunately, the angle θ is uncertain, and is uniformly distributed across the interval $0 < \theta < \pi$.

(a) [10] Find the state operator ρ. Your answer should look something like $\rho = \sum_{n,m} a_{n,m}|n\rangle\langle m|$. Check that $\mathrm{Tr}(\rho) = \sum_n \langle n|\rho|n\rangle = 1$.

If we knew the phase θ, we would have $\rho = |\psi\rangle\langle\psi|$. However, since we don't, we have to take this combination and average it over all phases in the relevant range, so we have

$$\begin{aligned}
\rho &= \int_0^\pi \frac{d\theta}{\pi}|\psi\rangle\langle\psi| = \int_0^\pi \frac{d\theta}{2\pi}\left(|7\rangle + e^{i\theta}|8\rangle\right)\left(\langle 7| + e^{-i\theta}\langle 8|\right) \\
&= \int_0^\pi \frac{d\theta}{2\pi}\left(|7\rangle\langle 7| + e^{i\theta}|8\rangle\langle 7| + e^{-i\theta}|7\rangle\langle 8| + |8\rangle\langle 8|\right) \\
&= \frac{1}{2\pi}\left(\theta|7\rangle\langle 7| - ie^{i\theta}|8\rangle\langle 7| + ie^{-i\theta}|7\rangle\langle 8| + \theta|8\rangle\langle 8|\right)_0^\pi \\
&= \frac{1}{2\pi}\left(\pi|7\rangle\langle 7| + 2i|8\rangle\langle 7| - 2i|7\rangle\langle 8| + \pi|8\rangle\langle 8|\right) \\
&= \frac{1}{2}\left(|7\rangle\langle 7| + |8\rangle\langle 8|\right) + \frac{i}{\pi}\left(|8\rangle\langle 7| - |7\rangle\langle 8|\right)
\end{aligned}$$

The trace is

$$\mathrm{Tr}(\rho) = \sum_n \langle n|\left[\frac{1}{2}\left(|7\rangle\langle 7| + |8\rangle\langle 8|\right) + \frac{i}{\pi}\left(|8\rangle\langle 7| - |7\rangle\langle 8|\right)\right]|n\rangle = \frac{1}{2} + \frac{1}{2} = 1.$$

(b) [7] Find the expectation value of the Hamiltonian *H* for this state operator.

We use the formula $\langle A\rangle = \mathrm{Tr}(\rho A)$ to give us

$$\begin{aligned}
\langle H\rangle &= \mathrm{Tr}(\rho H) = \hbar\omega\sum_n \langle n|\rho\left(a^\dagger a + \tfrac{1}{2}\right)|n\rangle = \hbar\omega\sum_n \langle n|\rho\left(n + \tfrac{1}{2}\right)|n\rangle \\
&= \hbar\omega\left(\left(7 + \tfrac{1}{2}\right)\langle 7|\rho|7\rangle + \left(8 + \tfrac{1}{2}\right)\langle 8|\rho|8\rangle\right) = \hbar\omega\left[\tfrac{1}{2}\left(7 + \tfrac{1}{2}\right) + \tfrac{1}{2}\left(8 + \tfrac{1}{2}\right)\right] = 8\hbar\omega.
\end{aligned}$$

(c) [8] Find the expectation value of the momentum *P* for this state operator.

We use the formula $\langle A\rangle = \mathrm{Tr}(\rho A)$ to give us

$$\begin{aligned}
\langle P\rangle &= \mathrm{Tr}(\rho P) = i\sqrt{\tfrac{1}{2}m\omega\hbar}\sum_n \langle n|\rho\left(a^\dagger - a\right)|n\rangle = i\sqrt{\tfrac{1}{2}m\omega\hbar}\sum_n \langle n|\rho\left(\sqrt{n+1}|n+1\rangle - \sqrt{n}|n-1\rangle\right) \\
&= i\sqrt{\tfrac{1}{2}m\omega\hbar}\left(\langle 7|\rho\sqrt{8}|8\rangle - \langle 8|\rho\sqrt{8}|7\rangle\right) = i\sqrt{4m\omega\hbar}\left(-\frac{i}{\pi} - \frac{i}{\pi}\right) = \frac{4}{\pi}\sqrt{m\omega\hbar}.
\end{aligned}$$

6. The Kernel or propagator for the Harmonic oscillator with potential $\frac{1}{2}m\omega^2 x^2$ is given by

$$K(x,t;x_0,t_0)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin[\omega(t-t_0)]}}\exp\left[\frac{im\omega}{2\hbar\sin[\omega(t-t_0)]}\left\{\cos[\omega(t-t_0)](x^2+x_0^2)-2xx_0\right\}\right].$$

(a) [7] From this, deduce the free propagator in the limit $\omega\to 0$. Simplify as much as possible.

We simply take the limit $\omega\to 0$, but must be a bit careful to not get zeros in the denominators. This may be accomplished by realizing that for small θ, we approximate $\sin\theta=\theta$ and $\cos\theta=1$, so we have

$$\begin{aligned}\lim_{\omega\to 0}K(x,t;x_0,t_0)&=\sqrt{\frac{m\omega}{2\pi i\hbar\omega(t-t_0)}}\exp\left[\frac{im\omega}{2\hbar\omega(t-t_0)}\left\{(x^2+x_0^2)-2xx_0\right\}\right]\\&=\sqrt{\frac{m}{2\pi i\hbar(t-t_0)}}\exp\left[\frac{im(x-x_0)^2}{2\hbar(t-t_0)}\right].\end{aligned}$$

(b) [18] At $t=0$, the wave function is given by $\psi(x,t=0)=(A/\pi)^{1/4}e^{-Ax^2/2}$. Find the wave function at $\omega t=\frac{1}{2}\pi$. Simplify as much as possible.

We first rewrite this as $\psi(x_0,t_0=0)=(A/\pi)^{1/4}e^{-Ax_0^2/2}$, and then we have

$$\begin{aligned}\psi(x,t)&=\int K(x,t;x_0,0)\psi(x_0,0)\,dx_0\\&=\left(\frac{A}{\pi}\right)^{1/4}\int\sqrt{\frac{m\omega}{2\pi i\hbar\sin[\omega t]}}\exp\left[\frac{im\omega}{2\hbar\sin[\omega t]}\left\{\cos[\omega t](x^2+x_0^2)-2xx_0\right\}\right]\exp\left(-\frac{1}{2}Ax_0^2\right)dx_0\,.\end{aligned}$$

We now set , $\omega t=\frac{1}{2}\pi$, which simplifies things considerably:

$$\psi(x,t=\pi/2\omega)=\left(\frac{A}{\pi}\right)^{1/4}\int\sqrt{\frac{m\omega}{2\pi i\hbar}}\exp\left[-\frac{im\omega}{2\hbar}2xx_0\right]\exp\left(-\frac{1}{2}Ax_0^2\right)dx_0\,.$$

This is an integral of the form given in the useful formulas, with $\alpha=A$ and $\beta=-im\omega x/\hbar$. So we have

$$\psi(x,t=\pi/2\omega)=\left(\frac{A}{\pi}\right)^{1/4}\sqrt{\frac{m\omega}{2\pi i\hbar}}\sqrt{\frac{2\pi}{A}}\exp\left[\frac{1}{2A}\left(\frac{im\omega x}{\hbar}\right)^2\right]=\sqrt{\frac{m\omega}{i\hbar\sqrt{\pi A}}}\exp\left[-\frac{m^2\omega^2x^2}{2A\hbar^2}\right].$$

Possibly Useful Formulas

1-D Harmonic Osc. $V = \frac{1}{2}m\omega^2 x^2$ $a\|n\rangle = \sqrt{n}\|n-1\rangle$ $a^\dagger\|n\rangle = \sqrt{n+1}\|n+1\rangle$ $X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$ $P = i\sqrt{\frac{1}{2}\hbar m\omega}\left(a^\dagger - a\right)$	1-D infinite square well, size L $E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2}$ State Operator $\rho = \sum_i f_i \|\psi_i\rangle\langle\psi_i\|$ $\langle A\rangle = \mathrm{Tr}(\rho A)$	Clebsch-Gordan coeff. $\langle j_1, j_2; m_1, m_2 \| j, m\rangle$ $\langle 1,\frac{1}{2};0,\frac{1}{2}\|\frac{3}{2},\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}$, $\langle 1,\frac{1}{2};0,\frac{1}{2}\|\frac{1}{2},\frac{1}{2}\rangle = -\sqrt{\frac{1}{3}}$, $\langle 1,\frac{1}{2};1,-\frac{1}{2}\|\frac{3}{2},\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}$, $\langle 1,\frac{1}{2};1,-\frac{1}{2}\|\frac{1}{2},\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}$. EM Fields $\mathbf{B} = \nabla\times\mathbf{A}$ $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla U$ EM Hamiltonian $H = \frac{\boldsymbol{\pi}^2}{2m} - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S}$ Propagator: $\Psi(x,t) = \int dx_0\, K(x,t;x_0,t_0)\,\Psi(x_0,t_0)$

Possibly Useful Integral: $\int_{-\infty}^{\infty} e^{-\alpha y^2/2+\beta y}\,dy = \sqrt{2\pi/\alpha}\,e^{\beta^2/2\alpha}$