Physics 741 – Graduate Quantum Mechanics 1

Solutions to Midterm Exam, Fall 2016

1. Consider the wave function $\psi(x)=\begin{cases} Nx(a-x) & 0<x<a\,, \\ 0 & \text{otherwise.}\end{cases}$

Once properly normalized, this wave function has $\langle X\rangle = \frac{1}{2}a$ **and** $\langle X^2\rangle = \frac{2}{7}a^2$**.**

(a) [5] What is the correct normalization *N*?

We insist that the normalization integral yields one, so we have

$$1=\int_{-\infty}^{\infty}\left|\psi(x)\right|^2 dx = N^2\int_0^a\left[x(a-x)\right]^2 dx = N^2\int_0^a\left(a^2x^2-2ax^3+x^4\right)dx$$
$$= N^2\left(\tfrac{1}{3}a^2x^3-\tfrac{1}{2}ax^4+\tfrac{1}{5}x^5\right)\Big|_0^a = N^2a^5\left(\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}\right)=\tfrac{1}{30}N^2a^5\,,$$
$$N=\sqrt{30/a^5}\,.$$

(b) [8] What are $\langle P\rangle$ and $\langle P^2\rangle$ for this state?

We simply insert the operator $P=-i\,d/dx$ to find

$$\langle P\rangle = -i\hbar\int_{-\infty}^{\infty}\psi^*(a)\frac{d}{dx}\psi(x)\,dx = N^2\int_0^a\left[x(a-x)\right]\frac{d}{dx}\left[x(a-x)\right]dx$$
$$=-i\hbar\frac{30}{a^5}\int_0^a\left(ax-x^2\right)(a-2x)\,dx = -i\hbar\frac{30}{a^5}\int_0^a\left(a^2x-3ax^2+2x^4\right)dx$$
$$=-i\hbar\frac{30}{a^5}\left(\tfrac{1}{2}a^2x^2-ax^3+\tfrac{1}{2}x^4\right)\Big|_0^a = -i\hbar\frac{30}{a}60\left(\tfrac{1}{2}-1+\tfrac{1}{2}\right)=0\,,$$
$$\langle P^2\rangle = -\hbar^2\int_{-\infty}^{\infty}\psi^*(a)\frac{d^2}{dx^2}\psi(x)\,dx = -\hbar^2\frac{30}{a^5}\int_0^a\left[x(a-x)\right]\frac{d^2}{dx^2}\left[x(a-x)\right]dx$$
$$=\frac{60\hbar^2}{a^5}\int_0^a\left(ax-x^2\right)dx = \frac{60\hbar^2}{a^5}\left(\tfrac{1}{2}ax^2-\tfrac{1}{3}x^3\right)\Big|_0^a = \frac{60\hbar^2}{a^3}\left(\tfrac{1}{2}-\tfrac{1}{3}\right)=\frac{10\hbar^2}{a^2}\,.$$

(c) [7] Find the uncertainties Δx and Δp and show that they satisfy the uncertainty relation.

$$\Delta x=\sqrt{\langle X^2\rangle-\langle X\rangle^2}=\sqrt{\tfrac{2}{7}a^2-\left(\tfrac{1}{2}a\right)^2}=a\sqrt{\tfrac{2}{7}-\tfrac{1}{4}}=a\sqrt{\tfrac{1}{28}}=\tfrac{1}{\sqrt{28}}a\,,$$
$$\Delta p=\sqrt{\langle P^2\rangle-\langle P\rangle^2}=\sqrt{\frac{10\hbar^2}{a^2}-0^2}=\frac{\sqrt{10}\hbar}{a}\,.$$

This yields $(\Delta x)(\Delta p)=\sqrt{\tfrac{5}{14}}\hbar = 0.578\hbar > \tfrac{1}{2}\hbar\,.$

2. A particle of mass *m* lies in the infinite square well with allowed region $0 < x < a$. The wave function takes the form $\psi(x) = \begin{cases} N\sin^2(\pi x/a) & 0 < x < a, \\ 0 & \text{elsewhere.} \end{cases}$

(a) [5] Determine the normalization constant *N*.

With the help of the helpful integrals, we have

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = N^2 \int_0^a \sin^4(\pi x/a)\, dx = N^2 \int_0^a \sin^2(\pi x/a)\sin^2(\pi x/a)\, dx = \tfrac{3}{8} N^2 a ,$$

$$N = \sqrt{8/3a} .$$

(b) [7] Write this state in the form $|\psi\rangle = \sum_n c_n |\phi_n\rangle$, where $|\phi_n\rangle$ are the energy eigenstates. Some helpful integrals are provided.

The normalized energy eigenstates and eigenvalues are given by

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right), \quad E_n = \frac{\pi^2 n^2 \hbar^2}{2ma^2} .$$

The overlap c_n are given by

$$c_n = \langle \phi_n | \psi \rangle = \sqrt{\frac{2}{a}} \sqrt{\frac{8}{3a}} \int_{-a}^{a} \sin\left(\frac{\pi n x}{a}\right) \sin^2\left(\frac{\pi x}{a}\right) dx = \frac{4}{\sqrt{3}a} \frac{4 \cdot 1^2 a}{\pi n (4 - n^2)} \quad \text{if } n \text{ odd, zero otherwise.}$$

Simplifying and substituting into the sum, we have

$$|\psi\rangle = \sum_{n \text{ odd}} \frac{16}{\pi n \sqrt{3}(4 - n^2)} |\phi_n\rangle .$$

(c) [8] If we were to measure the energy, what would be the possible outcomes and corresponding probabilities? Give a general formula, and find the numeric value as a percentage for the first three non-zero outcomes.

The energies were given above, namely $E_n = \pi^2 n^2 \hbar^2 / 2ma^2$, but the probability vanishes unless *n* is odd. For *n* odd, we have

$$P(n) = |\langle \phi_n | \psi \rangle|^2 = |c_n|^2 = \frac{256}{3\pi^2 n^2 (n^2 - 4)^2} .$$

The table at right gives the resulting probabilities for the first three non-zero cases. Note that the probabilities add to 99.99%. They should total one, which doubtless just represents the contribution from larger *n*.

n	E_n	$P(n)$
1	$\frac{\pi^2\hbar^2}{2ma^2}$	96.07%
3	$\frac{9\pi^2\hbar^2}{2ma^2}$	3.84%
5	$\frac{25\pi^2\hbar^2}{2ma^2}$	0.08%

3. Consider the harmonic oscillator with mass *m* and angular frequency ω. At *t* = 0, the system is in the state $\left|\Psi(t=0)\right\rangle = N\sum_{n=1}^{\infty}\frac{i^n}{n^2}\left|n\right\rangle$.

(a) [7] What is the correct normalization *N*? Some helpful sums are given on the next page.

We need to have

$$1=\left\langle\Psi\middle|\Psi\right\rangle = N^2\sum_{p=1}^{\infty}\frac{(-i)^p}{p^2}\left\langle p\right|\sum_{n=1}^{\infty}\frac{i^n}{n^2}\left|n\right\rangle = N^2\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-i)^p i^n}{p^2n^2}\delta_{np} = N^2\sum_{n=1}^{\infty}\frac{1}{n^4} = N^2\zeta(4) = \frac{\pi^4N^2}{90},$$

$$N=\sqrt{90}/\pi^2 .$$

(b) [5] Find the value of $\left\langle P\right\rangle$ for this state. Simplify as much as possible.

We write the operators in terms of raising and lowering operators, so we have

$$\left\langle P\right\rangle = \left\langle\Psi\right|P\left|\Psi\right\rangle = \frac{90}{\pi^4}i\sqrt{\frac{1}{2}\hbar m\omega}\left[\sum_{p=1}^{\infty}\frac{(-i)^p}{p^2}\left\langle p\right|\right]\left(a^\dagger - a\right)\left[\sum_{n=1}^{\infty}\frac{i^n}{n^2}\left|n\right\rangle\right]$$

$$=\frac{45}{\pi^4}i\sqrt{2\hbar m\omega}\left[\sum_{p=1}^{\infty}\frac{(-i)^p}{p^2}\left\langle p\right|\right]\left[\sum_{n=1}^{\infty}\frac{i^n}{n^2}\left(\sqrt{n+1}\left|n+1\right\rangle - \sqrt{n}\left|n-1\right\rangle\right)\right]$$

$$=\frac{45}{\pi^4}i\sqrt{2\hbar m\omega}\left[\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{i^n(-i)^p}{n^2p^2}\left(\sqrt{n+1}\delta_{n+1,p} - \sqrt{n}\delta_{n-1,p}\right)\right]$$

The smart way to simplify this is to use the delta function to do the *p*-sum on the first term and to do the *n* – sum on the second term. Then we have

$$\left\langle P\right\rangle = \frac{45}{\pi^4}i\sqrt{2\hbar m\omega}\left[\sum_{n=1}^{\infty}\frac{i^n(-i)^{n+1}}{n^2(n+1)^2}\sqrt{n+1} - \sum_{p=1}^{\infty}\frac{i^{p+1}(-i)^p}{p^2(p+1)^2}\sqrt{p+1}\right]$$

$$=\frac{45}{\pi^4}\sqrt{2\hbar m\omega}\left[\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)^{3/2}} + \sum_{p=1}^{\infty}\frac{1}{p^2(p+1)^{3/2}}\right] = \frac{90}{\pi^4}\sqrt{2\hbar m\omega}\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)^{3/2}}.$$

Other than numerically, I don't know of any way to simplify this further.

(c) [8] What is $\left|\Psi(t)\right\rangle$ at all times?

Each of the eigenstates has energy $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$, so when we include time-dependance, they simply pick up a factor of $\exp\left(-iE_nt/\hbar\right) = \exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right]$. So the time state vector is

$$\left|\Psi(t)\right\rangle = \sum_{n=1}^{\infty}c_ne^{-i\left(n+\frac{1}{2}\right)\omega t}\left|n\right\rangle = \frac{3\sqrt{10}}{\pi^2}\sum_{n=1}^{\infty}\frac{i^n}{n^2}e^{-i\left(n+\frac{1}{2}\right)\omega t}\left|n\right\rangle.$$

4. A hydrogen atom is in the state $\left|n,l,m\right\rangle = \left|2,1,0\right\rangle$.

(a) [6] What would be the result if you measure the energy, orbital angular momentum squared L^2 and *z*-component L_z?

Because we are in an eigenstate of all three quantities, the three requested quantities are given by

$$E = -\frac{13.6\text{ eV}}{n^2} = -\frac{13.6\text{ eV}}{2^2} = -3.40\text{ eV},$$
$$\mathbf{L}^2 = \hbar^2\left(l^2 + l\right) = \hbar^2\left(1^2 + 1\right) = 2\hbar^2,$$
$$L_z = \hbar m = 0.$$

(b) [6] Write the explicit form of the wave function $\psi\left(r,\theta,\phi\right)$.

We simply write it down using

$$\psi_{nlm}\left(r,\theta,\phi\right) = R_{nl}\left(r\right)Y_l^m\left(\theta,\phi\right) = R_{21}\left(r\right)Y_1^0\left(\theta,\phi\right) = \frac{re^{-r/2a}}{2\sqrt{6a^5}}\frac{\sqrt{3}}{2\sqrt{\pi}}\cos\theta = \frac{re^{-r/2a}}{4\sqrt{2\pi a^5}}\cos\theta.$$

(c) [8] Calculate the expectation value $\left\langle R^{-1}\right\rangle$ for this wave function, where *R* is the distance from the origin operator.

We can save some steps using the fact that the spherical harmonics are orthonormal when integrated over angles, so we have

$$\left\langle R^{-1}\right\rangle = \int \psi_{210}^*\left(\mathbf{r}\right)r^{-1}\psi_{210}\left(\mathbf{r}\right)d^3\mathbf{r} = \int_0^\infty \left[R_{21}\left(r\right)\right]^2 r^2 r^{-1}dr\int Y_1^0\left(\theta,\phi\right)^* Y_1^0\left(\theta,\phi\right)d\Omega$$
$$= \int_0^\infty \left[\frac{re^{-r/2a}}{2\sqrt{6a^5}}\right]^2 rdr = \frac{1}{24a^5}\int_0^\infty r^3 e^{-r/a}dr = \frac{1}{24a^5}a^4 3! = \frac{1}{4a}.$$

5. In a certain basis, the state vector is given by $|\Psi\rangle = \begin{pmatrix} \frac{1}{3}+\frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}$, and the spin operator in the *x*-directions is given by $S_x = \frac{1}{2}\hbar\sigma_x = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(a) [10] Find the eigenvalues and normalized eigenvectors of S_x.

We first find the eigenvalues and eigenvectors of the Pauli matrix σ_x, which are found from

$$0 = \det(\sigma_x - \lambda\mathbf{1}) = \det\begin{pmatrix} 0-\lambda & 1 \\ 1 & \lambda \end{pmatrix} = \lambda^2 - 1,$$
$$\lambda = \pm 1.$$

The eigenvalues for S_x will then be $\pm\frac{1}{2}\hbar$. We can then find the eigenvectors by giving them arbitrary components, and solving the eigenvector equation, so we have

$$\pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}, \quad \text{so} \quad \beta = \pm\alpha, \quad |\pm\tfrac{1}{2}\hbar\rangle = \begin{pmatrix} \alpha \\ \pm\alpha \end{pmatrix}.$$

Normalizing them, we find $2\alpha^2 = 1$, so $\alpha = 1/\sqrt{2}$, and we have'

$$|\pm\tfrac{1}{2}\hbar\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$

(b) [10] If you measured S_x, what is the probability that you get each of the possible eigenvalues you found in part (a)? What would be the state vector afterwards?

The probabilities are given by

$$P(+\tfrac{1}{2}\hbar) = \left|\langle +\tfrac{1}{2}\hbar|\psi\rangle\right|^2 = \frac{1}{2}\left|\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3}+\frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}\right|^2 = \frac{1}{2}\left|\tfrac{1}{3}+\tfrac{2}{3}i+\tfrac{2}{3}\right|^2 = \tfrac{1}{2}\left[1+\left(\tfrac{2}{3}\right)^2\right] = \tfrac{13}{18},$$
$$P(-\tfrac{1}{2}\hbar) = \left|\langle -\tfrac{1}{2}\hbar|\psi\rangle\right|^2 = \frac{1}{2}\left|\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{3}+\frac{2}{3}i \\ \frac{2}{3} \end{pmatrix}\right|^2 = \frac{1}{2}\left|\tfrac{1}{3}+\tfrac{2}{3}i-\tfrac{2}{3}\right|^2 = \tfrac{1}{2}\left[\left(-\tfrac{1}{3}\right)^2+\left(\tfrac{2}{3}\right)^2\right] = \tfrac{5}{18}.$$

Since there are no states with degenerate eigenvalues, you must (up to a phase) end up in these eigenstates, so in the first case you will end up in the state $|+\frac{1}{2}\hbar\rangle$ and in the latter case $|-\frac{1}{2}\hbar\rangle$.

Possibly Helpful Formulas:

Harmonic Oscillator	Radial Wave Functions	Spherical Harmonics	Hydrogen Energy
$X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$	$R_{10}(r) = \frac{2e^{-r/a_0}}{\sqrt{a^3}}$	$Y_1^0(\theta,\phi) = \frac{\sqrt{3}}{2\sqrt{\pi}}\cos\theta$	$E = -\frac{13.6\text{ eV}}{n^2}$
$P = i\sqrt{\tfrac{1}{2}\hbar m\omega}\left(a^\dagger - a\right)$	$R_{20}(r) = \frac{e^{-r/2a}}{\sqrt{2a^3}}\left(1 - \frac{r}{2a}\right)$	$Y_2^0(\theta,\phi) = \frac{\sqrt{5}}{4\sqrt{\pi}}\left(3\cos^2\theta - 1\right)$	
$a\lvert n\rangle = \sqrt{n}\lvert n-1\rangle$			
$a^\dagger\lvert n\rangle = \sqrt{n+1}\lvert n+1\rangle$	$R_{21}(r) = \frac{re^{-r/2a}}{2\sqrt{6a^5}}$	$Y_2^{\pm 1}(\theta,\phi) = \mp\frac{\sqrt{15}\,e^{\pm i\phi}}{2\sqrt{2\pi}}\sin\theta\cos\theta$	

Possibly Helpful Integrals:

Definite Integrals: *n* and *p* are assumed to be positive integers

$$\int_0^\infty x^n e^{-\alpha x}dx = \frac{n!}{\alpha^{n+1}}, \qquad \int_0^a \sin\left(\frac{\pi n x}{a}\right)dx = \begin{cases} \frac{2a}{\pi n} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases} \qquad \int_0^a \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi p x}{a}\right)dx = \tfrac{1}{2}a\delta_{np}.$$

$$\int_0^a \sin\left(\frac{\pi n x}{a}\right)\sin^2\left(\frac{\pi p x}{a}\right)dx = \begin{cases} \frac{4p^2 a}{\pi n\left(4p^2 - n^2\right)} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases} \qquad \int_0^a \sin^2\left(\frac{\pi n x}{a}\right)\sin^2\left(\frac{\pi p x}{a}\right)dx = a\left(\tfrac{1}{4} + \tfrac{1}{8}\delta_{np}\right).$$

Possibly helpful sums:

$$\sum_{n=1}^\infty \frac{1}{n^k} = \zeta(k), \quad \sum_{n=1}^\infty \frac{(-1)^k}{n^k} = \left(2^{1-k} - 1\right)\zeta(k), \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$