Physics 741 – Graduate Quantum Mechanics 1

Solutions to Midterm Exam, Fall 2014

Please note that some possibly helpful formulas and integrals appear on the second page. The number of points for each question is marked at the start, with points for each part marked separately.

1. [20 points] Consider the wave function $\psi(x) = N(x+ia)^{-2}$. This state, once properly normalized, has expectation values $\langle P\rangle = \frac{3}{2}\hbar a^{-1}$ and $\langle P^2\rangle = 3\hbar^2 a^{-2}$.

(a) [7] What is the correct normalization *N*?

We demand that the normalization integral yield 1, so we have

$$1 = \int_{-\infty}^{\infty} \psi^* \psi \, dx = \int_{-\infty}^{\infty} \frac{N^2 dx}{(x-ia)^2 (x+ia)^2} = \int_{-\infty}^{\infty} \frac{N^2 dx}{(x^2+a^2)^2} = \frac{\pi N^2}{2a^3},$$

$$N^2 = 2a^3/\pi , \quad N = \sqrt{2a^3/\pi} .$$

(b) [7] What is $\langle X\rangle$ and $\langle X^2\rangle$ for this state?

We simply compute

$$\langle X\rangle = \int_{-\infty}^{\infty} \psi^* x \psi \, dx = \int_{-\infty}^{\infty} \frac{N^2 x dx}{(a^2+x^2)^2} = \frac{2a^3}{\pi}\cdot 0 = 0 ,$$

$$\langle X^2\rangle = \int_{-\infty}^{\infty} \psi^* x^2 \psi \, dx = \int_{-\infty}^{\infty} \frac{N^2 x^2 dx}{(a^2+x^2)^2} = \frac{2a^3}{\pi}\cdot\frac{\pi}{2a} = a^2 .$$

(c) [6] Find the uncertainties Δx and Δp and show that they satisfy the uncertainty relation.

The uncertainties are computed using

$$(\Delta x)^2 = \langle X^2\rangle - \langle X\rangle^2 = a^2 - 0 = a^2 ,$$

$$(\Delta p)^2 = \langle P^2\rangle - \langle P\rangle^2 = \frac{3\hbar^2}{a^2} - \left(\frac{3\hbar}{2a}\right)^2 = \frac{\hbar^2}{a^2}\left(3 - \frac{9}{4}\right) = \frac{3\hbar^2}{4a^2},$$

$$\Delta x = a , \quad \Delta p = \frac{\hbar\sqrt{3}}{2a}, \quad \Delta x \, \Delta p = \tfrac{\sqrt{3}}{2}\hbar > \tfrac{1}{2}\hbar .$$

2. [20 points] A particle of mass *m* lies in the infinite square well with allowed region $0 < x < a$. At *t* = 0, the wave function takes the form $\Psi(x,t=0) = N\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{5\pi x}{a}\right)$ in the allowed region and it vanishes elsewhere

(a) [7] Write this state in the form $|\Psi(0)\rangle = \sum_n c_n |\phi_n\rangle$. Some helpful integrals are provided.

We need to find the c_n's, which are given by

$$c_n = \langle\phi_n|\Psi(0)\rangle = \int \phi_n^*(x)\Psi(x,0)\,dx = N\sqrt{\frac{2}{a}}\int_0^a \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{5\pi x}{a}\right)dx$$

$$= N\sqrt{\frac{2}{a}}\frac{a}{4}\left(\delta_{n,2+5} + \delta_{2,5+n} - \delta_{5,n+2}\right) = N\frac{\sqrt{2a}}{4}\left(\delta_{n,7} - \delta_{3,n}\right)$$

The middle term doesn't contribute because we *n* is positive. In summary,

$$c_3 = -\tfrac{1}{4}N\sqrt{2a}\,, \quad c_7 = \tfrac{1}{4}N\sqrt{2a}\,, \quad \text{all others vanish.}$$

(b) [6] Determine the normalization constant *N* such that $|\Psi\rangle$ is normalized.

To make it normalized, we need

$$1 = \langle\Psi(0)|\Psi(0)\rangle = \sum_n \langle\Psi(0)|\phi_n\rangle\langle\phi_n|\Psi(0)\rangle = \sum_n c_n^* c_n = \left(\tfrac{1}{4}N\right)^2(2a+2a) = \tfrac{1}{4}N^2 a\,,$$

$$N = 2\big/\sqrt{a}$$

It follows that

$$c_3 = -\frac{1}{4}\frac{2}{\sqrt{a}}\sqrt{2a} = \frac{-1}{\sqrt{2}}\,, \quad c_7 = \frac{1}{4}\frac{2}{\sqrt{a}}\sqrt{2a} = \frac{1}{\sqrt{2}}\,, \quad \text{all others vanish.}$$

(c) [7] Write $|\Psi(t)\rangle$ as a function of time in terms of the eigenstate basis, and write $\Psi(x,t)$.

We have

$$|\Psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar}|\phi_n\rangle = c_3\exp\left(-\frac{i9\pi^2\hbar t}{2ma^2}\right)|\phi_3\rangle + c_7\exp\left(-\frac{i49\pi^2\hbar t}{2ma^2}\right)|\phi_7\rangle$$

$$= \frac{1}{\sqrt{2}}\left[-\exp\left(-\frac{i9\pi^2\hbar t}{2ma^2}\right)|\phi_3\rangle + \exp\left(-\frac{i49\pi^2\hbar t}{2ma^2}\right)|\phi_7\rangle\right]$$

$$\Psi(x,t) = \langle x|\Psi(t)\rangle = \frac{1}{\sqrt{a}}\left[-\exp\left(-\frac{i9\pi^2\hbar t}{2ma^2}\right)\sin\left(\frac{3\pi x}{a}\right) + \exp\left(-\frac{i49\pi^2\hbar t}{2ma^2}\right)\sin\left(\frac{7\pi x}{a}\right)\right].$$

3. [25 points] In a certain basis, the Hamiltonian takes the form $H = \hbar\omega\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$**.**

(a) [12] Find the eigenvalues and <u>normalized</u> eigenvectors of this Hamiltonian.

To find the eigenvalues, we first factor out the common factor of $\hbar\omega$ and then use the characteristic equation of the remaining matrix:

$$0 = \det\begin{pmatrix} 1-\lambda & \sqrt{2} \\ \sqrt{2} & 0-\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

This has roots of $\lambda = 2$ and $\lambda = -1$, so the original eigenvalues are $2\hbar\omega$ and $-\hbar\omega$. To find the eigenvectors, we return to the reduced matrix, and we have

$$2\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + \beta\sqrt{2} \\ \alpha\sqrt{2} \end{pmatrix} \quad \text{or} \quad -1\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + \beta\sqrt{2} \\ \alpha\sqrt{2} \end{pmatrix}.$$

Equating these pair of expressions in either case we have

$$\begin{Bmatrix} 2\alpha = \alpha + \beta\sqrt{2} \\ 2\beta = \alpha\sqrt{2} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} -\alpha = \alpha + \beta\sqrt{2} \\ -\beta = \alpha\sqrt{2} \end{Bmatrix}.$$

Each pair of equations is, in fact, redundant, and simply reduces to

$$\alpha = \beta\sqrt{2} \quad \text{or} \quad \beta = -\alpha\sqrt{2}.$$

Up to normalization, our vectors are now

$$|2\hbar\omega\rangle = \begin{pmatrix} \beta\sqrt{2} \\ \beta \end{pmatrix} \quad \text{and} \quad |-\hbar\omega\rangle = \begin{pmatrix} \alpha \\ -\alpha\sqrt{2} \end{pmatrix}.$$

If we demand that these be normalized, we get in the first case the equation $3\beta^2 = 1$ and in the second $3\alpha^2 = 1$. Hence the normalized eigenvectors are (up to arbitrary phase)

$$|2\hbar\omega\rangle = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} \quad \text{and} \quad |-\hbar\omega\rangle = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}.$$

(b) [7] At *t* = 0, the state is in the state $|\Psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$**. Find** $|\Psi(t)\rangle$ **at all times.**

The general solution is $|\Psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar}|\phi_n\rangle$ where $c_n = \langle\phi_n|\Psi(t=0)\rangle$. It is then evident that $c_2 = \sqrt{\frac{2}{3}}$ and $c_{-1} = \sqrt{\frac{1}{3}}$. We therefore have

$$\left|\Psi(t)\right\rangle = c_2 e^{-i2\hbar\omega t/\hbar}\left|2\hbar\omega\right\rangle + c_2 e^{i\hbar\omega t/\hbar}\left|-\hbar\omega\right\rangle = \sqrt{\tfrac{2}{3}}e^{-2i\omega t}\begin{pmatrix}\sqrt{\frac{2}{3}}\\ \sqrt{\frac{1}{3}}\end{pmatrix} + \sqrt{\tfrac{1}{3}}e^{i\omega t}\begin{pmatrix}\sqrt{\frac{1}{3}}\\ -\sqrt{\frac{2}{3}}\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2e^{-2i\omega t}+e^{i\omega t}\\ \sqrt{2}\left(e^{-2i\omega t}-e^{i\omega t}\right)\end{pmatrix}.$$

(c) [6] At time *t*, if we were to measure the energy, what would be the possible outcomes and corresponding probabilities?

The energy can take on only one of the two eigenvalues, so the only possibilities are

$$P(2\hbar\omega) = \left|\left\langle\phi_2\middle|\Psi(t)\right\rangle\right|^2 = \left|\sqrt{\tfrac{2}{3}}e^{-2i\omega t}\right|^2 = \tfrac{2}{3},\quad P(-\hbar\omega) = \left|\left\langle\phi_{-1}\middle|\Psi(t)\right\rangle\right|^2 = \left|\sqrt{\tfrac{1}{3}}e^{i\omega t}\right|^2 = \tfrac{1}{3}.$$

4. [15 points] Consider the harmonic oscillator with mass *m* and angular frequency ω.

(a) [7] For which non-negative integers *q* will the matrix elements $\langle 47|X^q|50\rangle$ or $\langle 47|P^q|50\rangle$ be non-zero? I want a complete rule that lets me tell when they are non-zero.

The operators X and P each can either increase or decrease n by exactly one. Hence to get from 50 to 47, we must decrease n by three, which can be achieved in three steps, so $q \geq 3$. Furthermore, if q is even, then if we start at n = 50, we must have changed by one an even number of times, and thus if q is even, the matrix element must vanish. Hence we must have $q \geq 3$ and q odd.

(b) [8] For the smallest *q* for which they do not vanish, compute them.

The smallest value is q = 3, for which we must select the operator a every time to get a non-vanishing matrix element. Hence we have

$$\begin{aligned}\langle 47|X^3|50\rangle &= \left(\frac{\hbar}{2m\omega}\right)^{3/2}\langle 47|\left(a+a^\dagger\right)^3|50\rangle = \left(\frac{\hbar}{2m\omega}\right)^{3/2}\langle 47|a^3|50\rangle \\ &= \left(\frac{\hbar}{2m\omega}\right)^{3/2}\sqrt{50\cdot 49\cdot 48}\,\langle 47|47\rangle = \left(\frac{\hbar}{2m\omega}\right)^{3/2} 5\cdot 7\cdot 4\sqrt{2\cdot 3} = 70\sqrt{\frac{3\hbar^3}{m^3\omega^3}}\,,\end{aligned}$$

$$\begin{aligned}\langle 47|P^3|50\rangle &= \left(\tfrac{1}{2}\hbar m\omega\right)^{3/2} i^3\langle 47|\left(a^\dagger-a\right)^3|50\rangle = \left(\tfrac{1}{2}\hbar m\omega\right)^{3/2} i\langle 47|a^3|50\rangle = \left(\tfrac{1}{2}\hbar m\omega\right)^{3/2} i5\cdot 7\cdot 4\sqrt{2\cdot 3} \\ &= 70i\sqrt{3\hbar^3 m^3\omega^3}\,.\end{aligned}$$

5. **[20 points] Bottomonium consists of a bottom quark and bottom anti-quark, each of mass m_b, bound by a potential that is approximately $V(r) = Ar$, where r is the distance between them.**

   (a) **[4] Find a formula for the reduced mass of this system in terms of the quark mass m_b.**

   The reduced mass is given by $\dfrac{1}{\mu} = \dfrac{1}{m_b} + \dfrac{1}{m_b} = \dfrac{2}{m_b} \quad \Rightarrow \quad \mu = \tfrac{1}{2} m_b$.

   (b) **[5] What is an appropriate choice of coordinates for this system? Name two operators that commute with each other and with the Hamiltonian. Label the eigenstates of H by their eigenvalues under these two new operators. What can you tell me about these eigenvalues?**

The problem is spherically symmetric, which means that all three of the angular momentum operators $\mathbf{L}$ commute with H. However, they don't commute with each other. But we know in general that $\mathbf{L}^2$ commutes with any of the L's, so we can pick our two operators to be $\mathbf{L}^2$ and L_z. We then label our states as $|n,l,m\rangle$, in such a way that

$$\mathbf{L}^2|n,l,m\rangle = \hbar^2\left(l^2+l\right)|n,l,m\rangle \quad \text{and} \quad L_z|n,l,m\rangle = \hbar m|n,l,m\rangle.$$

The eigenvalue l must be a non-negative integer (l = 0,1,2, etc.) and m must be an integer whose magnitude is no larger than l ($m = -l,\ -l+1, -l+2, \ldots, l$).

   (c) **[5] Factor the wave function into an angular and a radial part. Describe completely on of these functions.**

For spherically symmetric problems, the general solution is $\psi(\mathbf{r}) = R(r)Y(\theta,\phi)$. The angular part will be spherical harmonics, $Y_l^m(\theta,\phi)$. These have the properties

$$\mathbf{L}^2 Y_l^m(\theta,\phi) = \hbar^2\left(l^2+l\right)Y_l^m(\theta,\phi) \quad \text{and} \quad L_z Y_l^m(\theta,\phi) = \hbar m Y_l^m(\theta,\phi).$$

   (d) **[6] For the remaining function, write an ordinary differential equation for the function. Do not attempt to solve it.**

We substitute it into Schrödinger's time independent equation to yield

$$\begin{aligned}
ER(r)Y_l^m(\theta,\phi) &= \frac{1}{2\mu}\mathbf{P}^2 R(r)Y_l^m(\theta,\phi) + V(r)R(r)Y_l^m(\theta,\phi) \\
&= \frac{-\hbar^2}{2\mu r}\frac{\partial^2}{\partial r^2}\left[rR(r)\right]Y_l^m(\theta,\phi) + \frac{1}{2\mu r^2}\mathbf{L}^2 Y_l^m(\theta,\phi)R(r) + ArR(r)Y_l^m(\theta,\phi)
\end{aligned}$$

$$ER(r) = \frac{-\hbar^2}{m_b r}\frac{d^2}{dr^2}\left[rR(r)\right] + \frac{\hbar^2\left(l^2+l\right)}{m_b r^2}R(r) + ArR(r).$$