Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 8

8.9 [10] The quadrupole operators are spherical tensors of rank 2; that is, a spherical tensor with $k = 2$. Its components are:

$$T_{\pm 2}^{(2)} = \tfrac{1}{2}\left(X \pm iY\right)^2, \quad T_{\pm 1}^{(2)} = \mp XZ - iYZ, \quad T_0^{(2)} = \sqrt{\tfrac{1}{6}}\left(2Z^2 - X^2 - Y^2\right)$$

(a) [2] Show that these operators either commute or anti-commute with parity, Π.

Parity anti-commutes with the operators X, Y, and Z, so we have, for example

$$\Pi T_{\pm 1}^{(2)} = \Pi\left(\mp XZ - iYZ\right) = \pm X\Pi Z + iY\Pi Z = \left(\mp XZ - iYZ\right)\Pi = T_{\pm 1}^{(2)}\Pi.$$

It is clear this method generalizes to any of the five operators, so $\Pi T_q^{(2)} = T_q^{(2)}\Pi$.

(b) [3] To calculate electric quadrupole radiation, it is necessary to calculate matrix elements of the form $\left\langle \alpha lm \middle| T_q^{(2)} \middle| \alpha' l' m' \right\rangle$. Based on the Wigner Eckart theorem, what constraints can we put on *m'*, *m*, and *q*? What constraints can we put on *l* and *l'*?

The Wigner-Eckart theorem tells us that $m = m' + q$ and l lies in the range $\left|l' - 2\right| \le l \le l' + 2$.

(c) [2] Based on parity, what constraints can we put on *l* and *l'*?

Take our equation showing that parity commutes with the electric quadrupole moments, and sandwich it between two states, and we have

$$\left\langle \alpha lm \middle| \Pi T_q^{(2)} \middle| \alpha' l' m' \right\rangle = \left\langle \alpha lm \middle| T_q^{(2)}\Pi \middle| \alpha' l' m' \right\rangle,$$

$$(-1)^l \left\langle \alpha lm \middle| T_q^{(2)} \middle| \alpha' l' m' \right\rangle = (-1)^{l'} \left\langle \alpha lm \middle| T_q^{(2)} \middle| \alpha' l' m' \right\rangle.$$

Assuming the matrix elements don't vanish, this can happen only if l and l' have the same parity, that is, they are both odd or both even.

(d)[3] Given *l'*, what values of *l* are acceptable? List all acceptable values of *l* for *l'* = 0, 1, 2, 3, 4, 5.

Well, since l is in the range $\left|l' - 2\right| \le l \le l' + 2$, then if l' is two or bigger, then this becomes $l' - 2 \le l \le l' + 2$. With the additional constraint that they be of the same parity, the only possible l values are $l' - 2$, l', and $l' + 2$. However, when l' is 1, the restriction becomes that $l = 1$ or 3, and for $l' = 0$, then only $l = 2$ is allowed. The table at right summarizes this in several cases.

l'	l
0	2
1	1,3
2	0,2,4
3	1,3,5
4	2,4,6
5	3,5,7

10. [15] Suppose the Hamiltonian takes the form $H = \mathbf{P}^2/(2m) + V(\mathbf{R}) + W(|\mathbf{R}|)(\mathbf{L}\cdot\mathbf{S})$, where V and W are arbitrary real functions, and L and S are the orbital angular momentum and spin operators. Show that if $\Psi(\mathbf{r},t)$ is a solution of Schrödinger's equation, then so is

(a) [5] $\Psi^*(\mathbf{r},-t)$ if the particle has no spin (so the spin term isn't there); and

We start with Schrödinger's equation in this case, which is

$$i\hbar\frac{\partial}{dt}\Psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t) + V(\mathbf{r})\Psi(\mathbf{r},t).$$

Taking the complex conjugate, this implies

$$-i\hbar\frac{\partial}{dt}\Psi^*(\mathbf{r},-t) = -\frac{\hbar^2}{2m}\nabla^2\Psi^*(\mathbf{r},-t) + V(\mathbf{r})\Psi^*(\mathbf{r},-t).$$

We now change the variable t to $-t$. Note that there is also a t in the derivative on the left, and this becomes

$$i\hbar\frac{\partial}{dt}\Psi^*(\mathbf{r},-t) = -\frac{\hbar^2}{2m}\nabla^2\Psi^*(\mathbf{r},-t) + V(\mathbf{r})\Psi^*(\mathbf{r},-t).$$

This is exactly what we wanted.

(b) [10] $-i\sigma_y\Psi^*(\mathbf{r},-t)$ for a spin ½ particle (so $\mathbf{S} = \frac{1}{2}\hbar\boldsymbol{\sigma}$).

The Schrödinger equation of course has a new term, so we add $W(r)(\mathbf{L}\cdot\mathbf{S})\Psi(\mathbf{r},t)$ at the end. We take the complex conjugate of this expression, which yields

$$-i\hbar\frac{\partial}{dt}\Psi^*(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\Psi^*(\mathbf{r},t) + V(\mathbf{r})\Psi^*(\mathbf{r},t) + W(r)(\mathbf{L}^*\cdot\mathbf{S}^*)\Psi^*(\mathbf{r},t).$$

Note that $\mathbf{L}\Psi = (\mathbf{R}\times\mathbf{P})\Psi = -i\hbar(\mathbf{r}\times\nabla)\Psi$, so we see that $\mathbf{L}^* = -\mathbf{L}$. Substituting this and then multiply by $-i\sigma_y$ on the left everywhere, we have

$$\begin{aligned}-i\hbar\frac{\partial}{dt}(-i\sigma_y)\Psi^*(\mathbf{r},t) &= -\frac{\hbar^2}{2m}\nabla^2(-i\sigma_y)\Psi^*(\mathbf{r},t) + V(\mathbf{r})(-i\sigma_y)\Psi^*(\mathbf{r},t)\\ &\quad - W(r)(-i\sigma_y)(\mathbf{L}\cdot\mathbf{S}^*)\Psi^*(\mathbf{r},t).\end{aligned}$$

Replacing $t \to -t$ as before, and writing out explicitly $\mathbf{S} = \frac{1}{2}\hbar\boldsymbol{\sigma}$, we have

$$\begin{aligned}i\hbar\frac{\partial}{dt}(-i\sigma_y)\Psi^*(\mathbf{r},-t) &= -\frac{\hbar^2}{2m}\nabla^2(-i\sigma_y)\Psi^*(\mathbf{r},-t) + V(\mathbf{r})(-i\sigma_y)\Psi^*(\mathbf{r},-t)\\ &\quad - \tfrac{1}{2}\hbar W(r)(-i\sigma_y)(\mathbf{L}\cdot\boldsymbol{\sigma}^*)\Psi^*(\mathbf{r},-t).\end{aligned}$$

Looking at the explicit form of the Pauli matrices, it is easy to take the complex conjugates to yield

$$-i\hbar\frac{\partial}{dt}\left(-i\sigma_y\right)\Psi^*\left(\mathbf{r},-t\right)=-\frac{\hbar^2}{2m}\nabla^2\left(-i\sigma_y\right)\Psi^*\left(\mathbf{r},-t\right)+V\left(\mathbf{r}\right)\left(-i\sigma_y\right)\Psi^*\left(\mathbf{r},-t\right)$$
$$-\tfrac{1}{2}\hbar W\left(r\right)\left(-i\sigma_y\right)\left(L_x\sigma_x-L_y\sigma_y+L_z\sigma_z\right)\Psi^*\left(\mathbf{r},-t\right).$$

Now the Pauli matrices have the property that they commute with themselves (of course) and anti-commute with each other, so $\sigma_y\sigma_x=-\sigma_x\sigma_y$ and $\sigma_y\sigma_z=-\sigma_z\sigma_y$. Substituting, we have

$$-i\hbar\frac{\partial}{dt}\left(-i\sigma_y\right)\Psi^*\left(\mathbf{r},t\right)=-\frac{\hbar^2}{2m}\nabla^2\left(-i\sigma_y\right)\Psi^*\left(\mathbf{r},t\right)+V\left(\mathbf{r}\right)\left(-i\sigma_y\right)\Psi^*\left(\mathbf{r},t\right)$$
$$+\tfrac{1}{2}\hbar W\left(r\right)\left(L_x\sigma_x+L_y\sigma_y+L_z\sigma_z\right)\left(-i\sigma_y\right)\Psi^*\left(\mathbf{r},t\right).$$

Now we just reconstruct the spin operator, and we have

$$-i\hbar\frac{\partial}{dt}\left(-i\sigma_y\right)\Psi^*\left(\mathbf{r},-t\right)=\left[-\frac{\hbar^2}{2m}\nabla^2+V\left(\mathbf{r}\right)+W\left(r\right)\left(\mathbf{L}\cdot\mathbf{S}\right)\right]\left(-i\sigma_y\right)\Psi^*\left(\mathbf{r},-t\right).$$