Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 7

7.1 [10] Derive Eqs. (7.10) using only the commutation relations Eqs. (7.7).

Eq. (7.10a) is trivial: $J_\pm^\dagger = \left(J_x \pm iJ_y\right)^\dagger = J_x \mp iJ_y = J_\mp$. For Eq. (7.10b), we just start working it out:

$$\begin{aligned}\left[\mathbf{J}^2, J_x\right] &= 0 + \left[J_y^2, J_x\right] + \left[J_z^2, J_x\right] = J_y\left[J_y, J_x\right] + \left[J_y, J_x\right]J_y + J_z\left[J_z, J_x\right] + \left[J_z, J_x\right]J_z \\ &= i\hbar\left(-J_yJ_z - J_zJ_y + J_zJ_y + J_yJ_z\right) = 0, \\ \left[\mathbf{J}^2, J_y\right] &= \left[J_x^2, J_y\right] + 0 + \left[J_z^2, J_y\right] = J_x\left[J_x, J_y\right] + \left[J_x, J_y\right]J_x + J_z\left[J_z, J_y\right] + \left[J_z, J_y\right]J_z \\ &= i\hbar\left(J_xJ_z + J_zJ_x - J_zJ_x - J_xJ_z\right) = 0, \\ \left[\mathbf{J}^2, J_z\right] &= \left[J_x^2, J_z\right] + \left[J_y^2, J_z\right] + 0 = J_x\left[J_x, J_z\right] + \left[J_x, J_z\right]J_x + J_y\left[J_y, J_z\right] + \left[J_y, J_z\right]J_y \\ &= i\hbar\left(-J_xJ_y - J_yJ_x + J_yJ_x + J_xJ_y\right) = 0.\end{aligned}$$

The remaining relations follow trivially: $\left[\mathbf{J}^2, J_\pm\right] = \left[\mathbf{J}^2, J_x \pm iJ_y\right] = \left[\mathbf{J}^2 J_x\right] \pm i\left[\mathbf{J}^2, J_y\right] = 0$.

For (7.10c) we again just work it out:

$$\left[J_z, J_\pm\right] = \left[J_z, J_x\right] \pm i\left[J_z, J_y\right] = i\hbar J_y \mp i^2\hbar J_x = \pm\hbar J_x + i\hbar J_y = \pm\hbar\left(J_x \pm i\hbar J_y\right) = \pm\hbar J_\pm.$$

And finally, for (7.10d), we expand the right side and show that it is equal to the left side:

$$\begin{aligned}J_\mp J_\pm + J_z^2 \pm \hbar J_z &= \left(J_x \mp iJ_y\right)\left(J_x \pm iJ_y\right) + J_z^2 \pm \hbar J_z = J_x^2 \pm iJ_xJ_y \mp iJ_yJ_x + J_y^2 + J_z^2 \pm \hbar J_z \\ &= J_x^2 + J_y^2 + J_z^2 \pm i\left[J_x, J_y\right] \pm \hbar J_z = J_x^2 + J_y^2 + J_z^2 \pm i^2\hbar J_z \pm \hbar J_z = \mathbf{J}^2\end{aligned}$$

7.2 [10] For j = 2, we will work out the explicit form for all of the matrices $\mathbf{J}$.

(a) [5] Write out the expression for J_z and $J_\pm$ as an appropriately sized matrix.

Since j = 2, the matrix will be of size $2\cdot 2 + 1 = 5$. For J_3, we will have a diagonal matrix with elements running from $2\hbar$ down to $-2\hbar$. For J_+, we will have elements just along the diagonal, where the value in the row labeled by m will be $\hbar\sqrt{j^2 + j - m^2 + m}$, and J_- is just the Hermitian conjugate of J_+. Hence we have

$$J_z = \hbar\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \quad J_+ = \hbar\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_- = \hbar\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

(b) [2] Write out J_x and J_y.

This is just a matter of taking $J_x = \left(J_+ + J_-\right)/2$ and $J_y = \left(J_+ - J_-\right)/2i$

$$J_x = \hbar\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},\quad J_y = \hbar\begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & -i\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & i\sqrt{\frac{3}{2}} & 0 & -i\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & i\sqrt{\frac{3}{2}} & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}$$

(c) [3] Check explicitly that $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$ is a constant matrix with the appropriate value.

$$J^2 = \hbar^2\begin{pmatrix} 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \frac{5}{2} & 0 & \frac{3}{2} & 0 \\ \sqrt{\frac{3}{2}} & 0 & 3 & 0 & \sqrt{\frac{3}{2}} \\ 0 & \frac{3}{2} & 0 & \frac{5}{2} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \end{pmatrix} + \hbar^2\begin{pmatrix} 1 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \frac{5}{2} & 0 & -\frac{3}{2} & 0 \\ -\sqrt{\frac{3}{2}} & 0 & 3 & 0 & -\sqrt{\frac{3}{2}} \\ 0 & -\frac{3}{2} & 0 & \frac{5}{2} & 0 \\ 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 1 \end{pmatrix} + \hbar^2\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

$$= \hbar^2\begin{pmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} = 6\hbar^2\mathbf{1}$$

The appropriate value is $\hbar^2\left(j^2 + j\right) = 6\hbar^2$.