Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 6

6.4 [15] A particle of mass M in *three* dimensions has potential $V(X,Y,Z)=\frac{1}{4}A\left(X^2+Y^2\right)^2$.

(a) [6] Show that this Hamiltonian has *two* continuous symmetries, and that they commute. Call the corresponding eigenvalues *m* and *k*. Are there any restrictions on *k* and *m*?

First, it is obvious that the potential is independent of Z, and therefore there is a continuous translation symmetry in this direction. Secondly, it is easy to see that rotation about the z-axis leaves the Hamiltonian unchanged. Specifically, define a set of rotated operators

$$X' = X\cos\theta - Y\sin\theta\,,$$
$$Y' = X\sin\theta + Y\cos\theta\,.$$

Then if we treat the potential as $V(x,y)=\frac{1}{4}A\left(x^2+y^2\right)^2$, then we have

$$V(X',Y')=\tfrac{1}{4}A\left[\left(X\cos\theta - Y\sin\theta\right)^2+\left(X\sin\theta+Y\cos\theta\right)^2\right]^2$$
$$=\tfrac{1}{4}A\begin{bmatrix} X^2\cos^2\theta - 2XY\cos\theta\sin\theta + Y^2\sin^2\theta \\ +X^2\sin^2\theta + 2XY\sin\theta\cos\theta + Y^2\cos^2\theta\end{bmatrix}=\tfrac{1}{4}A\left(X^2+Y^2\right)^2=V(X,Y).$$

Because we have translation symmetry in the z-direction and rotation about the z-axis, our Hamiltonian will commute with the generators of these groups, P_z and L_z. Our energy eigenstates can also be chosen to be eigenstates of these operators, and we will have

$$P_z\left|\phi\right\rangle = \hbar k\left|\phi\right\rangle \quad\text{and}\quad L_z\left|\phi\right\rangle=\hbar m\left|\phi\right\rangle.$$

As argued in class, the eigenvalue m is forced to be an integer, though k is unrestricted.

(b) [9] What would be an appropriate set of coordinates for writing the eigenstates of this Hamiltonian? Write the eigenstates as a product of three functions (which I call Z, R, and Φ), and give me the explicit form of two of these functions.

Clearly, z is a good coordinate to use, since our eigenstates of the Hamiltonian are eigenstates of P_z. However, since they are also eigenstates of L_z, it seems like a good idea to change coordinates to cylindrical coordinates (ρ,ϕ,z), which are related to Cartesian coordinates by

$$\begin{Bmatrix} x=\rho\cos\phi \\ y=\rho\sin\phi \\ z=z\end{Bmatrix}\quad\text{OR}\quad\begin{Bmatrix}\rho=\sqrt{x^2+y^2} \\ \phi=\tan^{-1}(y/x) \\ z=z\end{Bmatrix}$$

If we write our wave function in terms of these coordinates, and assume it factors, we have

$$\psi(\rho,\phi,z) = R(\rho)\Phi(\phi)Z(z)$$

If we demand that this be an eigenstate of P_z with eigenvalue $\hbar k$, then we find

$$\hbar k Z(z) = P_z Z(z) = \frac{\hbar}{i}\frac{\partial}{\partial z} Z(z) \quad \text{so that} \quad Z(z) = e^{ikz} .$$

Similarly, if we demand that $\psi(\rho,\phi,z)$ be an eigenstate of L_z with eigenvalue $\hbar m$, then we find

$$\hbar m \Phi(\phi) = L_z \Phi(\phi) = \frac{\hbar}{i}\frac{\partial}{\partial \phi}\Phi(\phi) \quad \text{so that} \quad \Phi(\phi) = e^{im\phi} .$$

There is a certain arbitrariness in normalization, and the choices we have made have perhaps not been the best, but up to a constant, we therefore find

$$\psi(\rho,\phi,z) = R(\rho) e^{ikz+im\phi} .$$

If we wished, we could now easily write an explicit equation for the radial function *R*. Writing the Laplacian that is implicit in the kinetic term in the Hamiltonian in cylindrical coordinates, we find

$$H\psi = -\frac{\hbar^2}{2M}\left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\right) + \tfrac{1}{4}A\left(\rho^2\right)^2\psi .$$

Plugging in our explicit form for the wave function, and using Schrödinger's equation $H\psi = E\psi$, we have

$$ER = -\frac{\hbar^2}{2M}\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right) + \left(\frac{\hbar^2k^2}{2M} + \frac{\hbar^2m^2}{2M\rho^2} + \tfrac{1}{4}A\rho^4\right)R .$$