Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 6

6.3 [20] A particle of mass *m* and energy *E* in two dimensions is incident on a plane step function given by

$$V(X,Y)=\begin{cases}0 & \text{if } X<0,\\ V_0 & \text{if } X>0.\end{cases}$$

The incoming wave has wave function $\psi_{\text{in}}(x,y)=e^{i(k_x x+k_y y)}$ **for *x* < 0.**

(a) [7] Write the Hamiltonian. Determine the energy *E* for the incident wave. Convince yourself that the Hamiltonian has a translation symmetry, and therefore that the transmitted and reflected wave will share something in common with the incident wave (they are all eigenstates of what operator?).

The Hamiltonian, of course, is just $H=P_x^2/2m+P_y^2/2m+V(X)$. Because the potential vanishes for the incoming wave, we have

$$H\psi_{\text{in}}=\frac{P_x^2+P_y^2}{2m}\psi_{\text{in}}=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi_{\text{in}}=\frac{\hbar^2}{2m}\left(k_x^2+k_y^2\right)\psi_{\text{in}},$$

so that $E=\hbar^2\left(k_x^2+k_y^2\right)/2m$. It is obvious that this Hamiltonian commutes with P_y, since the potential doesn't contain *Y*, and the kinetic part contains only momentum. It follows that eigenstates of the Hamiltonian can be chosen to also be eigenstates of P_y, and will have the same eigenvalue as the incoming wave $P_y|\psi\rangle=\hbar k_y|\psi\rangle$. This tells us the *y*-dependance will be the same for the incident, reflected, and transmitted wave. The eigenstates will therefore take the form

$$\psi(x,y)=e^{ik_y y}\chi(x).$$

It remains only to find the function $\chi(x)$.

(b) [7] Write the general form of the reflected and transmitted wave. Use Schrödinger's equation to solve for the values of the unknown parts of the momentum for each of these waves (assume $k_x^2\hbar^2/2m>V_0$).

If we plug our general solution into Schrödinger's equation, we have

$$Ee^{ik_y y}\chi(x)=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(e^{ik_y y}\chi(x)\right)+V(x)e^{ik_y y}\chi(x),$$

$$\frac{\hbar^2\left(k_x^2+k_y^2\right)}{2m}e^{ik_y y}\chi(x)=\left(\frac{\hbar^2k_y^2}{2m}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)e^{ik_y y}\chi(x)+V(x)e^{ik_y y}\chi(x).$$

The exponentials cancel everywhere, and some factors can be cancelled. Solving for the second derivative term, we find

$$\frac{\partial^2 \chi(x)}{\partial x^2} = \left[\frac{2mV(x,y)}{\hbar^2} - k_x^2\right]\chi(x).$$

This equation is easy to solve. For $x < 0$, the potential is zero, and we are solving $\chi'' = -k_x^2 \chi$ and the solutions are proportional to $e^{\pm ik_x x}$. For $x > 0$, we can define $k_x'^2 = k_x^2 - 2mV_0/\hbar^2$, and we are solving $\chi'' = -k_x'^2 \chi$ and the solutions are proportional to $e^{\pm ik_x' x}$.

The incident wave is $\chi_I = e^{+ik_x x}$. The wave going the other way in this region must be the reflected wave, $\chi_R \propto e^{-ik_x x}$. In the other region, the wave proportional to $e^{+ik_x' x}$ is a wave traveling to the right, so we have $\chi_T \propto e^{+ik_x' x}$. The wave proportional to $e^{-ik_x' x}$ would represent a wave moving to the left from infinity on the right, so this doesn't represent anything in this problem. Reinstating the y-dependence, and throwing in some constants of proportionality, our waves are

$$\psi_I = e^{+ik_x x + ik_y y}, \quad \psi_R = Be^{-ik_x x + ik_y y}, \quad \psi_T = Ce^{ik_x' x + ik_y y}.$$

(c) [6] Assume the wave function and its derivative are continuous across the boundary $x = 0$. Find the amplitudes for the transmitted and reflected waves, and find the probability R of the wave being reflected.

The wave in the region $x < 0$ is given by $\psi_I + \psi_R$, and on the right, by ψ_T. Matching these wave functions and their derivatives in the x-direction at the boundary, we have

$$e^{ik_y y} + Be^{ik_y y} = Ce^{ik_y y} \quad \text{and} \quad k_x e^{ik_y y} - Bk_x e^{ik_y y} = Ck_x' e^{ik_y y}$$

When we cancel the common phase on both sides of each of these equations, the first equation becomes $A + B = C$, and substituting this into the second yields

$$k_x - Bk_x = (1+B)k_x'.$$

Rearranging this a bit, we have

$$k_x - k_x' = B(k_x + k_x'), \quad B = \frac{k_x - k_x'}{k_x + k_x'} \quad \text{and} \quad C = 1 + B = \frac{2k_x}{k_x + k_x'}$$

The probability of reflection is the ratio of the amplitude squared for the reflected wave vs. the incident wave, so

$$R = |B|^2 = \left(\frac{k_x - k_x'}{k_x + k_x'}\right)^2 = \left(\frac{k_x - \sqrt{k_x^2 - 2mV_0/\hbar^2}}{k_x + \sqrt{k_x^2 - 2mV_0/\hbar^2}}\right)^2 ..$$