Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 4

4.1 [25] It is sometimes said that a watched kettle never boils. In some sense, this is true in quantum mechanics. Consider a quantum system where the state space is two dimensional, with basis states $\{|0\rangle,|1\rangle\}$, the former representing the kettle in the "not boiled" state, the latter the "boiled" state. In terms of these, the Hamiltonian is given by

$$H=\hbar\omega\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

(a) [6] At $t = 0$, the quantum state is given by $|\Psi(t=0)\rangle=|0\rangle=\begin{pmatrix}1\\0\end{pmatrix}$. Solve Schrödinger's equation given the initial conditions, and determine the ket of the kettle at later times.

Schrödinger's equation says that

$$i\hbar\frac{d}{dt}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}=H\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}=\hbar\omega\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}=i\hbar\omega\begin{pmatrix}-\beta(t)\\ \alpha(t)\end{pmatrix},$$

Or, $\dot\alpha(t)=-\omega\beta(t)$ and $\dot\beta(t)=\omega\alpha(t)$. These equations aren't that hard to solve by inspection, but one way to get it more directly is by taking another time derivative of the first equation, which yields $\ddot\alpha(t)=-\omega\dot b(t)=-\omega^2\alpha(t)$, which has general solution

$$\alpha(t)=A\cos(\omega t)+B\sin(\omega t)$$

The boundary conditions that $\alpha(0)=1$ and $\dot\alpha(0)=-\omega\beta(0)=0$ tells us $A = 1$ and $B = 0$. We then use $\dot\alpha(t)=-\omega\beta(t)$ to show that $\beta(t)=-\dot\alpha(t)/\omega=\sin(\omega t)$, and we have

$$|\Psi(t)\rangle=\cos(\omega t)|0\rangle+\sin(\omega t)|1\rangle$$

(b)[3] At time $t = \Delta t$, an observer checks whether the kettle has boiled yet. That is, he measures the quantum system using the boiled operator B, defined by $B|0\rangle=0|0\rangle$ and $B|1\rangle=1|1\rangle$. What is the probability P_1 that the kettle has boiled at this time (*i.e.*, that measuring B yields the eigenvalue 1)? If the kettle is boiled at this time, the total time $T = \Delta t$ is recorded.

The probability that it has boiled is

$$P_1=P(1)=\left|\langle 1|\Psi(t)\rangle\right|^2=\sin^2(\omega\Delta t).$$

For future reference, we will note also that the probability it has *not* boiled is $\cos^2(\omega\Delta t)$.

(c) [3] If the kettle is *not* boiled, what is the quantum state immediately *after* the measurement has been made?

According to postulate 5, the quantum state after a measurement of not boiled is

$$\left|\Psi(t^+)\right\rangle = |0\rangle\langle 0|\Psi(t)\rangle \Big/ \sqrt{P(0)} = |0\rangle\cos(\omega\Delta t)\Big/\sqrt{\cos^2(\omega\Delta t)} = |0\rangle .$$

In other words, it is back in its original state. Since there is only one not-boiled state, we actually knew this from the start.

(d) [3] After a second interval of Δt, the kettle is measured again to see if it has boiled. What is the probability P_2 that it is not boiled the first time, and it is boiled the second? If this occurs, the total time $T = 2\Delta t$ is recorded.

Since it ends up back in the original state, the probability after an equal interval must be exactly the same. The probability that it did not boil the first time, and it did boil the second, is

$$P_2 = (1-P_1)P_1 = \cos^2(\omega\Delta t)\sin^2(\omega\Delta t).$$

(e) [5] The process is repeated until the kettle has actually boiled. What is the general formula P_n that it first boils on the n'th measurement? Write a formula for the average time $\langle T\rangle = \langle n\Delta t\rangle$ that it takes for the kettle to boil. The formula below may be helpful.

The computation is similar. To first boil on the n'th trial, we must have $n-1$ failures, followed by success, so

$$P_n = (1-P_1)^{n-1}P_1 = \cos^{2n-2}(\omega\Delta t)\sin^2(\omega\Delta t)$$

The average time is just the sum of the products of the probabilities and the time in each case, so

$$\begin{aligned}\langle T\rangle &= \sum_{n=1}^{\infty} P_n n\Delta t = \Delta t\sin^2(\omega\Delta t)\sum_{n=1}^{\infty} n\cos^{2n-2}(\omega\Delta t)\\ &= \Delta t\sin^2(\omega\Delta t)\left[1+2\cos^2(\omega\Delta t)+3\cos^4(\omega\Delta t)+\cdots\right]\end{aligned}$$

The formula inside the square brackets is identical with the formula below if we set $x=\cos^2(\omega\Delta t)$, so we have

$$\langle T\rangle = \frac{\Delta t\sin^2(\omega\Delta t)}{\left[1-\cos^2(\omega\Delta t)\right]^2} = \frac{\Delta t\sin^2(\omega\Delta t)}{\sin^4(\omega\Delta t)} = \frac{\Delta t}{\sin^2(\omega\Delta t)}$$

(f) [2] Demonstrate that in the limit $\Delta t \to 0$, it takes forever for the kettle to boil.

$$\lim_{\Delta t \to 0} \langle T \rangle = \lim_{\Delta t \to 0} \left[\frac{\Delta t}{\sin^2(\omega \Delta t)} \right] = \lim_{\Delta t \to 0} \frac{\Delta t}{\omega^2 (\Delta t)^2} = \infty$$

(g) [3] Determine (numerically or otherwise) the optimal time Δt so that $\langle T \rangle = \langle n \Delta t \rangle$ will be minimized, so the kettle boils as quickly as possible.

The function will be minimized when the derivative vanishes, that is, when

$$0 = \frac{d}{d(\Delta t)} \langle T \rangle = \frac{1}{\sin^2(\omega \Delta t)} - \frac{2\omega \Delta t \cos(\omega \Delta t)}{\sin^3(\omega \Delta t)} = \frac{1 - 2(\omega \Delta t)\cot(\omega \Delta t)}{\sin^2(\omega \Delta t)}$$

If we let $x = \omega \Delta t$, we are trying to find a root of $1 - 2x\cot(x)$. We can search for the solution of this equation with the help of a calculator, or with the help of Maple

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> fsolve(tan(x)=2*x,x,1..2);
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Maple tells us the solution is $x = 1.16556$, so $\Delta t_{\text{min}} = 1.16556/\omega$.

Helpful formula: $1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) = \frac{1}{(1-x)^2}$