Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 3

3.3 [10] Prove the following identities about the operators *A*, *B*, and *C*:

(a) [4] Commutators of products:

$$[A,BC]=B[A,C]+[A,B]C \quad \text{and} \quad [AB,C]=A[B,C]+[A,C]B$$

You simply write out the right side explicitly in each case and then simplify it to give the left side.

$$B[A,C]+[A,B]C = BAC - BCA + ABC - BAC = ABC - BCA = [A,BC],$$
$$A[B,C]+[A,C]B = ABC - ACB + ACB - CAB = ABC - CAB = [AB,C].$$

(b) [4] The Jacobi identities:

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0 \quad \text{and} \quad [[A,B],C]+[[B,C],A]+[[C,A],B]=0$$

You simply write everything out explicitly in each case, and you find everything cancels beautifully:

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=[A,BC-CB]+[B,CA-AC]+[C,AB-BA]$$
$$= ABC - BCA - ACB - CBA + BCA - CAB - BAC + ACB + CAB - ABC - CBA + BAC = 0,$$
$$[[A,B],C]+[[B,C],A]+[[C,A],B]=[AB-BA,C]+[BC-CB,A]+[CA-AC,B]$$
$$= ABC - CAB - BAC + CBA + BCA - ABC - CBA + ACB + CAB - BCA - ACB + BAC = 0.$$

(c) [2] Show that if *A* and *B* are Hermitian, then $i[A,B]$ is also Hermitian

$$\{i[A,B]\}^\dagger = \{iAB - iBA\}^\dagger = B^\dagger A^\dagger(-i) - A^\dagger B^\dagger(-i) = iAB - iBA = i[A,B]$$

3.4 [10] Define the angular momentum operators $\mathbf{L} = \mathbf{R} \times \mathbf{P}$, or in components

$$L_x = YP_z - ZP_y\,, \qquad L_y = ZP_x - XP_z\,, \qquad L_z = XP_y - YP_x$$

(a) [6] With the help of problem 3(a), work out the six commutators $\left[L_{z,}\mathbf{R}\right]$ and $\left[L_{z,}\mathbf{P}\right]$.

$$\left[L_z, X\right] = \left[XP_y - YP_x, X\right] = X\left[P_y, X\right] + \left[X, X\right]P_y - Y\left[P_x, X\right] - \left[Y, X\right]P_x = -Y\left(-i\hbar\right) = i\hbar Y\,,$$

$$\left[L_z, Y\right] = \left[XP_y - YP_x, Y\right] = X\left[P_y, Y\right] + \left[X, Y\right]P_y - Y\left[P_x, Y\right] - \left[Y, Y\right]P_x = X\left(-i\hbar\right) = -i\hbar X\,,$$

$$\left[L_z, Z\right] = \left[XP_y - YP_x, Z\right] = X\left[P_y, Z\right] + \left[X, Z\right]P_y - Y\left[P_x, Z\right] - \left[Y, Z\right]P_x = 0\,,$$

$$\left[L_z, P_x\right] = \left[XP_y - YP_x, P_x\right] = X\left[P_y, P_x\right] + \left[X, P_x\right]P_y - Y\left[P_x, P_x\right] - \left[Y, P_x\right]P_x = \left(i\hbar\right)P_y = i\hbar P_y\,,$$

$$\left[L_z, P_y\right] = \left[XP_y - YP_x, P_y\right] = X\left[P_y, P_y\right] + \left[X, P_y\right]P_y - Y\left[P_x, P_y\right] - \left[Y, P_y\right]P_x = -\left(i\hbar\right)P_x$$
$$= -i\hbar P_x\,,$$

$$\left[L_z, P_z\right] = \left[XP_y - YP_x, P_z\right] = X\left[P_y, P_z\right] + \left[X, P_z\right]P_y - Y\left[P_x, P_z\right] - \left[Y, P_z\right]P_x = 0\,.$$

(b) [4] With the help of problems 3(a) and 4(a), work out the commutators $\left[L_{z,}L_x\right]$ and $\left[L_{z,}L_y\right]$.

We simply continue as before:

$$\begin{aligned}\left[L_z, L_x\right] &= \left[L_z, YP_z - ZP_y\right] = \left[L_z, Y\right]P_z + Y\left[L_z, P_z\right] - \left[L_z, Z\right]P_y - Z\left[L_z, P_y\right] \\ &= -i\hbar XP_z + 0 - 0 - Z\left(-i\hbar P_x\right) = i\hbar\left(ZP_x - XP_z\right) = i\hbar L_y\,, \\ \left[L_z, L_y\right] &= \left[L_z, ZP_x - XP_z\right] = \left[L_z, Z\right]P_x + Z\left[L_z, P_x\right] - \left[L_z, X\right]P_z - X\left[L_z, P_z\right] \\ &= 0 + Zi\hbar P_y - i\hbar YP_z - 0 = i\hbar\left(ZP_y - YP_z\right) = -i\hbar L_x\,.\end{aligned}$$