Physics 742 – Graduate Quantum Mechanics 2

Solutions to Chapter 15

4. [10] A particle of mass *m* is in the ground state $|1\rangle$ of an infinite square well with allowed region $0 < X < a$. To this potential is added a harmonic perturbation $W(t) = AX\cos(\omega t)$, where *A* is small.

(a) [6] Calculate the transition rate $\Gamma(1 \to n)$ for a transition to another level. Don't let the presence of a delta function bother you. What angular frequency ω is necessary to make the transition occur to level *n* = 2?

We first rewrite this perturbation in the form $W(t) = \frac{1}{2}AX\left(e^{i\omega t} + e^{-i\omega t}\right)$. Our interaction is thus $W = \frac{1}{2}AX$, and the matrix elements we require are $W_{1n} = \frac{1}{2}A\langle n|X|1\rangle$. The eigenstates and energies of the unperturbed square well are

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right), \quad E_n = \frac{\pi^2 n^2 \hbar^2}{2ma^2}.$$

Maple is happy to do the integrals for us. We find the matrix elements we want are

$$W_{n1} = \frac{A}{2}\langle n|X|1\rangle = \frac{A}{2}\cdot\frac{2}{a}\int_0^a x\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi x}{a}\right)dx = -\frac{2Aan}{\pi^2\left(n^2-1\right)^2}\left[1+(-1)^n\right].$$

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> assume(n::integer);
> integrate(A/a*sin(Pi*n*x/a)*sin(Pi*x/a)*x,x=0..a);
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The rate then is given by

$$\Gamma(1 \to n) = \frac{2\pi}{\hbar}\left|W_{n1}\right|^2 \delta\left(E_n - E_1 - \hbar\omega\right) = \frac{8a^2A^2n^2\left[1+(-1)^n\right]^2}{\pi^3\hbar\left(n^2-1\right)^4}\delta\left(\frac{\pi^2\hbar^2}{2ma^2}\left(n^2-1\right) - \hbar\omega\right).$$

For the transition to *n* = 2, we need the delta function to vanish, which requires a frequency of $\omega = 3\pi^2\hbar/2ma^2$.

(b) [4] Now, instead of keeping a constant frequency, the frequency is tuned continuously, so that at *t* = 0 the frequency is 0, and it rises linearly so that at *t* = *T* it has increased to the value $\omega(T) = 2\pi^2\hbar/ma^2$. The tuning is so slow that at any given time, we may treat it as a harmonic source. Argue that only the *n* = 2 state can become populated (to leading order in *A*). Calculate the probability of a transition using $P(1 \to 2) = \int_0^T \Gamma(1 \to 2)\,dt$.

The maximum frequency achieved is 4/3 times larger than required to make this transition, enough for the 1 to 2 transition, but not enough for anything higher. We therefore need only consider the transition to *n* = 2. The frequency as a function of time is

$$\omega(t) = \frac{2\pi^2\hbar t}{ma^2 T}$$

To find the total transition probability, we then simply integrate the transition rate

$$P(1\to 2) = \int_0^T \Gamma(1\to 2)\,dt = \frac{128a^2A^2}{81\pi^3\hbar}\int_0^T \delta\left(\frac{3\pi^2\hbar^2}{2ma^2} - \frac{2\pi^2\hbar^2 t}{ma^2 T}\right)dt = \frac{128a^2A^2}{81\pi^3\hbar}\frac{ma^2T}{2\pi^2\hbar^2} = \frac{64ma^4A^2T}{81\pi^5\hbar^3}.$$