

Physics 742 – Graduate Quantum Mechanics 2
Solutions to Chapter 15

3. [15] A particle of mass m is initially in the ground state of a harmonic oscillator with frequency ω . At time $t = 0$, a perturbation is suddenly turned on of the form $W(t) = AXe^{-\lambda t}$. At late times ($t \rightarrow \infty$), the quantum state is measured again.
- (a) [10] Calculate, to second order in A , the amplitude S_{n0} that it ends up in the state $|n\rangle$, for all n (most of them will be zero).

We will need matrix elements of the form $\langle n|W(t)|m\rangle = A\langle n|X|m\rangle e^{-\lambda t}$, which are given by

$$Ae^{-\lambda t} \langle n|X|m\rangle = Ae^{-\lambda t} \sqrt{\frac{\hbar}{2m\omega}} \langle n|(a + a^\dagger)|m\rangle = Ae^{-\lambda t} \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{m}\delta_{n,m-1} + \sqrt{n}\delta_{m,n-1}).$$

To second order, then, our amplitudes are

$$\begin{aligned} S_{00} &= \delta_{00} + \frac{1}{i\hbar} \int_0^\infty \langle 0|W(t)|0\rangle dt + \frac{1}{(i\hbar)^2} \sum_n \int_0^\infty \langle 0|W(t)|n\rangle e^{i\omega_n t} dt \int_0^t \langle n|W(t')|0\rangle e^{i\omega_n t'} dt' \\ &= 1 - \frac{A^2}{\hbar^2} \left(\sqrt{\frac{\hbar}{2m\omega}} \right)^2 \int_0^\infty e^{-\lambda t} e^{-i\omega t} dt \int_0^t e^{-\lambda t'} e^{i\omega t'} dt' = 1 - \frac{A^2}{2m\omega\hbar(-\lambda + i\omega)} \int_0^\infty e^{-\lambda t} e^{-i\omega t} [e^{-(\lambda-i\omega)t} - 1] dt \\ &= 1 + \frac{A^2}{4m\omega^2\hbar(\lambda - i\omega)} \left[\frac{1}{2\lambda} - \frac{1}{\lambda + i\omega} \right] = 1 - \frac{A^2(\lambda - i\omega)}{4m\omega\hbar\lambda(\lambda^2 + \omega^2)}, \end{aligned}$$

$$\begin{aligned} S_{10} &= \delta_{10} + \frac{1}{i\hbar} \int_0^\infty \langle 1|W(t)|0\rangle e^{i\omega_1 t} dt + \frac{1}{(i\hbar)^2} \sum_n \int_0^\infty \langle 1|W(t)|n\rangle e^{i\omega_n t} dt \int_0^t \langle n|W(t')|0\rangle e^{i\omega_n t'} dt' \\ &= \frac{A}{i\hbar} \sqrt{\frac{\hbar}{2m\omega}} \int_0^\infty e^{i\omega t - \lambda t} dt = \frac{A}{i\sqrt{2m\hbar\omega}(\lambda - i\omega)}, \end{aligned}$$

$$\begin{aligned} S_{20} &= \delta_{20} + \frac{1}{i\hbar} \int_0^\infty \langle 2|W(t)|0\rangle e^{i\omega_2 t} dt + \frac{1}{(i\hbar)^2} \sum_n \int_0^\infty \langle 2|W(t)|n\rangle e^{i\omega_n t} dt \int_0^t \langle n|W(t')|0\rangle e^{i\omega_n t'} dt' \\ &= -\frac{A^2}{\hbar^2} \left(\sqrt{\frac{\hbar}{2m\omega}} \right)^2 \int_0^\infty \sqrt{2} e^{i\omega t - \lambda t} dt \int_0^t e^{i\omega t' - \lambda t'} dt' = -\frac{A^2}{\sqrt{2}\hbar m\omega(i\omega - \lambda)} \int_0^\infty e^{i\omega t - \lambda t} [e^{i\omega t - \lambda t} - 1] dt \\ &= \frac{A^2}{\sqrt{2}\hbar m\omega(\lambda - i\omega)} \left[\frac{1}{2(\lambda - i\omega)} - \frac{1}{\lambda - i\omega} \right] = -\frac{A^2}{2\sqrt{2}\hbar m\omega(\lambda - i\omega)^2}, \end{aligned}$$

$$S_{n0} = 0, \quad n > 2.$$

(b) [5] Calculate, to at least second order, the probability that it ends up in the state $|n\rangle$. Check that the sum of the probabilities is 1, to second order in A .

We now simply square the amplitudes we found previously, and we find

$$P(0 \rightarrow 0) = |S_{00}|^2 = \left[1 - \frac{A^2 (\lambda - i\omega)}{4m\omega\hbar\lambda(\lambda^2 + \omega^2)} \right] \left[1 - \frac{A^2 (\lambda + i\omega)}{4m\omega\hbar\lambda(\lambda^2 + \omega^2)} \right] \approx 1 - \frac{A^2}{2m\omega\hbar(\lambda^2 + \omega^2)},$$

$$P(0 \rightarrow 1) = |S_{10}|^2 = \frac{A^2}{2m\hbar\omega(\lambda^2 + \omega^2)},$$

$$P(0 \rightarrow 2) = |S_{20}|^2 = \frac{A^4}{8\hbar^2 m^2 \omega^2 (\lambda^2 + \omega^2)^2}.$$

To second order in A , the first two expressions add to 1 and the third is zero.