Solutions to Chapter 14

6. [25] A particle of mass *m* scatters from a potential $V(r) = F\delta(r-a)$, so that the potential exists only at the surface of a thin sphere of radius *a*.

(a) [4] What equation must the radial wave functions $R_l(r)$ satisfy? Solve this equation in the regions *r* < *a* and *r* > *a*.

The radial wave functions must satisfy

$$\frac{1}{r}\frac{d^2}{dr^2}\left[rR_l(r)\right] = \left[\frac{l^2+l}{r^2} + U(r) - k^2\right]R_l(r) = \left[\frac{l^2+l}{r^2} + \frac{2mF}{\hbar^2}\delta(r-a) - k^2\right]R_l(r).$$

Away from the point *r* = *a*, the problem is simply that of a free particle, and the solution was worked out in class. The answer is spherical Bessel functions, and take the form

$$R_l(r) = \begin{cases} \gamma j_l(kr) - \delta n_l(kr) & r < a, \\ \alpha j_l(kr) - \beta n_l(kr) & r > a. \end{cases}$$

The constants will generally be different in the different regions.

(b) [6] Apply appropriate boundary at *r* = 0. Deduce appropriate boundary conditions at *r* = *a*.

We want the radial function to be well-behaved at *r* = 0, which implies we only want the well-behaved $j_l(r)$. Hence we demand $\delta = 0$. At the boundary *r* = *a*, we must take our radial Schrödinger equation and integrate it across the boundary. We first multiply both sides by *r*, then we have

$$\int_{a-\varepsilon}^{a+\varepsilon} \frac{d^2}{dr^2}\left[rR_l(r)\right]dr = \int_{a-\varepsilon}^{a+\varepsilon}\left[\frac{l^2+l}{r} + \frac{2mF}{\hbar^2}r\delta(r-a) - k^2 r\right]R_l(r)\,dr,$$

$$\left[rR_l(r)\right]'\Big|_{a-\varepsilon}^{a+\varepsilon} = \frac{2mF}{\hbar^2}aR_l(a).$$

Since the first derivative has a finite discontinuity, it follows that the wave function will be continuous at the boundary. This yields two boundary conditions on the wave function:

$$\alpha j_l(ka) - \beta n_l(ka) = \gamma j_l(ka),$$

$$\alpha\left[rj_l(kr)\right]'_{r=a} - \beta\left[rn_l(kr)\right]'_{r=a} = \frac{2mF}{\hbar^2}a\gamma j_l(ka) + \gamma\left[rj_l(kr)\right]'_{r=a}$$

Our goal, ultimately, will be to eliminate γ from these equations and deduce the ratio of β to α.

(c) [8] Assume now that $ka \ll 1$, so that the scattering will be dominated by the *l* = 0 term. Find a formula for the phase shift δ_0. Find the differential cross-section. Check that your formula agrees with the formula found in section C for the hard sphere in the case $F \to \infty$.

We substitute in the explicit form for *l* = 0, namely $j_0(x) = \sin x/x$ and $n_0(x) = -\cos x/x$. Then our two boundary conditions become

$$\alpha \sin(ka)/(ka) + \beta \cos(ka)/(ka) = \gamma \sin(ka)/(ka),$$
$$\alpha\left[\sin(kr)/k\right]'_{r=a} + \beta\left[\cos(kr)/k\right]'_{r=a} = \frac{2mF}{\hbar^2} a\gamma \sin(ka)/(ka) + \gamma\left[\sin(kr)/k\right]'_{r=a}.$$

Clear the fractions from the first of these and work out the derivatives in the second.

$$\alpha \sin(ka) + \beta \cos(ka) = \gamma \sin(ka),$$
$$\alpha \cos(ka) - \beta \sin(ka) = \frac{2mF}{\hbar^2 k}\gamma \sin(ka) + \gamma \cos(ka).$$

Cross multiply these and then cancel the common factor of γ.

$$\left[\alpha \cos(ka) - \beta \sin(ka)\right]\sin(ka) = \left[2mF \sin(ka)/\hbar^2 k + \cos(ka)\right]\left[\alpha \sin(ka) + \beta \cos(ka)\right],$$
$$\beta\left[-\hbar^2 k \sin^2(ka) - \hbar^2 k \cos^2(ka) - 2mF \sin(ka)\cos(ka)\right] = \alpha 2mF \sin^2(ka),$$
$$\frac{\beta}{\alpha} = \frac{-2mF \sin^2(ka)}{\hbar^2 k + 2mF \sin(ka)\cos(ka)}.$$

We hence have the phase shift

$$\tan\delta_0 = \frac{\beta}{\alpha} = \frac{-2mF \sin^2(ka)}{\hbar^2 k + 2mF \sin(ka)\cos(ka)} \approx \frac{-2mFk^2a^2}{\hbar^2 k + 2mFka} = \frac{-2mFa^2k}{\hbar^2 + 2mFa}.$$

where we used $ka \ll 1$ to approximate $\sin ka = ka$ and $\cos ka = 1$. It's then easy to see that the numerator is much smaller than the denominator, so we can also approximate $\sin\delta_0 = \tan\delta_0$, and hence the differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{4\pi}{k^2}\sin^2\delta_0 \left|Y_0^0\right|^2 = \frac{1}{k^2}\left(\frac{-2mFka^2}{\hbar^2 + 2mFa}\right)^2 = \frac{4m^2F^2a^4}{\left(\hbar^2 + 2mFa\right)^2}.$$

In the limit of infinite potential, we ignore the first term in the denominator compared to the second, so $d\sigma/d\Omega = a^2$, the same as we found before.

(d) [7] Redo the problem using the first Born approximation. Again assume $ka \ll 1$ (effectively, this means *Ka* = 0). Check that the resulting differential cross-section in this case is identical with that found above in the limit $F \to 0$.

In the first Born approximation, the differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3\mathbf{r}V(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\right|^2 \approx \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3\mathbf{r}V(\mathbf{r})\right|^2 = \frac{m^2}{4\pi^2\hbar^4}\left[4\pi\int_o^\infty r^2 dr F\delta(r-a)\right]^2$$

$$= \frac{m^2}{4\pi^2\hbar^4}\left(4\pi a^2 F\right)^2 = \frac{4m^2a^4F^2}{\hbar^4}.$$

In the limit of small *F*, the previous computation yields $d\sigma/d\Omega = 4m^2F^2a^4/\hbar^4$, so they match.