Physics 742 – Graduate Quantum Mechanics 1

Solutions to Chapter 13

1. [20] In section A we worked out the effect of the perturbation of the finite nuclear size assuming the nucleus is a uniform sphere with radius *a*.

(a) [6] Redo the calculation assuming instead that the nucleus is a spherical shell of radius *a* with total charge *e*.

Outside the nucleus, the electric field, and hence the potential, will be unchanged, so that we must have $U(r) = k_e e/r$ for $r > a$. Inside, the potential must be constant. To make it continuous, it must therefore have the value

$$U(r) = \begin{cases} k_e e/r & \text{if } r > a, \\ k_e e/a & \text{if } r < a. \end{cases}$$

Normally, we assume that the potential is equal to $k_e e/r$ everywhere. Multiplying by the electric charge – e, we see that the *difference* between the correct potential and the assumed potential is given by

$$W(r) = \begin{cases} 0 & \text{if } r > a, \\ k_e e^2/r - k_e e^2/a & \text{if } r < a. \end{cases}$$

We must now find the change in the energy level of a given hydrogen wave function. As in class, we will assume that the nucleus is so small that we can treat the electron wave function as constant over the tiny nucleus. So we find

$$\langle n,l,m|W|n,l,m\rangle = |\psi_{nlm}(0)|^2 \int W(r)\, d^3\mathbf{r}\,.$$

Of course, only the s-wave contributes, so we have

$$\varepsilon'_{n00} = \frac{1}{4\pi} R_{n0}^2(0) k_e e^2 \int_0^a 4\pi r^2 \left(r^{-1} - a^{-1}\right) dr = R_{n0}^2(0) k_e e^2 \left(\tfrac{1}{2}a^2 - \tfrac{1}{3}a^2\right) = \tfrac{1}{6} R_{n0}^2(0) k_e e^2 a^2\,.$$

(b) [6] Now, assume that the nucleus consists of two shells of radii a_1 and a_2, with charges q_1 and q_2 with $e = q_1 + q_2$. What is the shift due to the finite nuclear size in this case?

In this case, assuming $a_1 < a_2$, there will be three regions of electric field:

$$E = \begin{cases} k_e e/r^2 & r > a_2, \\ k_e q_1/r^2 & a_1 < r < a_2\,, \\ 0 & r < a_1. \end{cases}$$

We can integrate this to get the potential:

$$U(r)=\begin{cases} k_e q_1/r+k_e q_2/r & \text{if } r>a_2, \\ k_e q_1/r+k_e q_2/a_2 & \text{if } a_1<r<a_2, \\ k_e q_1/a_1+k_e q_2/a_2 & \text{if } r<a_1. \end{cases}$$

Multiplying by the electron charge, and subtracting the naive potential $-ke^2/r$, we find

$$W(r)=\begin{cases} 0 & \text{if } r>a_2, \\ k_e e\left(q_2/r-q_2/a_2\right) & \text{if } a_1<r<a_2, \\ k_e e\left(e/r-q_1/a_1-q_2/a_2\right) & \text{if } r<a_1. \end{cases}$$

To finish this part of the problem most efficiently, we note that

$$W(r)=W_1(r)+W_2(r) \quad \text{where} \quad W_i(r)=\begin{cases} 0 & \text{if } r>a_i\ , \\ k_e e q_i\left(r^{-1}-a_i^{-1}\right) & \text{if } r<a_i\ . \end{cases}$$

This makes sense since the change in potential from moving each of the pieces of the charge out to some radius a_i must simply add to make the total change in the potential. Clearly, the expectation value of W is just the sum of the two separate expectation values, and by comparison, these are so similar to part (a) that we can immediately write down the total effect.

$$\Delta E_{n00}=\tfrac{1}{6}R_{n0}^2(0)k_e e\left(q_1 a_1^2+q_2 a_2^2\right).$$

(c) [4] Generalize part (b) to the case of N charges of magnitude q_i at radii a_i. Now generalize to the case of continuous charge distribution $q(r)$ which is normalized so $\int q(r)\,dr=e$. No proof is needed, but if you did part (b) correctly, the generalization should be obvious.

The generalization is straightforward. For several charges totaling e, the perturbation is $W(r)=\sum_i W_i(r)$, where $W_i(r)$ is given in part (b). The resulting energy shift is

$$\Delta E_{n00}=\tfrac{1}{6}R_{n0}^2(0)k_e e\sum_i q_i a_i^2$$

If we have charges laid out continuously at different radii, the sum becomes an integral

$$\Delta E_{n00}=\tfrac{1}{6}R_{n0}^2(0)k_e e\int q(r)r^2\,dr$$

(d) [4] For a uniform sphere of charge q and radius a, what is the charge density ρ? What is the charge $q(r)dr$ in a thin spherical shell of radius r and thickness dr? Use this to find the energy shift for a uniform sphere. Check it against the formula derived in class.

The charge density is $\rho=e\big/\left(\tfrac{4}{3}\pi a^3\right)$. The charge between r and $r+dr$ is simply this times the area of a sphere of radius r, times its thickness, so

$$q(r)\,dr = 4\pi r^2 \rho\, dr = \frac{3er^2}{a^3}\, dr\,.$$

We now can simply find the total perturbation using the formula from part (c):

$$\Delta E_{n00} = \frac{1}{6} R_{n0}^2(0)\, k_e e \int_0^R \frac{3er^2}{a^3} r^2\, dr = \frac{1}{6} R_{n0}^2(0)\, k_e e^2 \frac{3}{5} a^2 = \frac{1}{10} k_e e^2 a^2 R_{n0}^2(0)\,.$$

This is identical to the class notes.