Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 11

7. [20] In lecture I showed how MWI can account for the results when you measure a series of electrons with spin states $|+_x\rangle$. In this problem, we will instead measure the spin of electrons with spin state $|\theta\rangle$, where $|\theta\rangle = \cos\left(\tfrac{1}{2}\theta\right)|+\rangle + \sin\left(\tfrac{1}{2}\theta\right)|-\rangle$.

(a) [4] According to the Copenhagen interpretation, if we measure the spin S_z of this particle, what results might we get, and what are the probabilities of each of these results? What is the average value we would get for this result?

The eigenvalues of S_z are $\pm\tfrac{1}{2}\hbar$, and the corresponding probabilities are simply

$$P\left(+\tfrac{1}{2}\hbar\right) = \left|\langle + | \theta \rangle\right| = \cos^2\left(\tfrac{1}{2}\theta\right) \quad \text{and} \quad P\left(-\tfrac{1}{2}\hbar\right) = \left|\langle - | \theta \rangle\right| = \sin^2\left(\tfrac{1}{2}\theta\right)$$

Therefore, the average value would be

$$\langle S_z \rangle = \cos^2\left(\tfrac{1}{2}\theta\right)\left(\tfrac{1}{2}\hbar\right) + \sin^2\left(\tfrac{1}{2}\theta\right)\left(-\tfrac{1}{2}\hbar\right) = \tfrac{1}{2}\hbar\left[\cos^2\left(\tfrac{1}{2}\theta\right) - \sin^2\left(\tfrac{1}{2}\theta\right)\right] = \tfrac{1}{2}\hbar\cos\theta$$

(b) [5] Now we start working on the Multiple Worlds calculation. Define the state of N particles as $|\Psi_N\rangle = |\theta\rangle \otimes |\theta\rangle \otimes \cdots \otimes |\theta\rangle$. Define the average spin operator $\bar{S}_z = \sum_{i=1}^{N} S_{Zi} \big/ N$. Write out explicitly $\bar{S}_z |\Psi_N\rangle$. You may find it useful to use the state $|-\theta\rangle = \cos\left(\tfrac{1}{2}\theta\right)|+\rangle - \sin\left(\tfrac{1}{2}\theta\right)|-\rangle$ in your answer.

It's helpful to first work out

$$S_z|\theta\rangle = S_z\left[\cos\left(\tfrac{1}{2}\theta\right)|+\rangle + \sin\left(\tfrac{1}{2}\theta\right)|-\rangle\right] = \tfrac{1}{2}\hbar\left[\cos\left(\tfrac{1}{2}\theta\right)|+\rangle - \sin\left(\tfrac{1}{2}\theta\right)|-\rangle\right] = \tfrac{1}{2}\hbar|-\theta\rangle$$

We therefore find

$$\bar{S}_z|\Psi_N\rangle = \frac{1}{N}\sum_{i=1}^{N} S_{Zi}|\theta,\theta\cdots\theta\rangle = \frac{\hbar}{2N}\left(|-\theta,\theta\cdots\theta\rangle + |\theta,-\theta\cdots\theta\rangle + \cdots + |\theta,\theta\cdots -\theta\rangle\right).$$

(c) [7] Work out the expectation values $\langle\psi|\bar{S}_z|\psi\rangle$ and $\langle\psi|\bar{S}_z^2|\psi\rangle$. I strongly recommend you first write out expressions for the four expressions $\langle\pm\theta|\pm\theta\rangle$.

We'll take the hint, and find

$$\langle\pm\theta|\pm\theta\rangle = \left(\cos\tfrac{\theta}{2}\langle+| \pm \sin\tfrac{\theta}{2}\langle-|\right)\left(\cos\tfrac{\theta}{2}|+\rangle \pm \sin\tfrac{\theta}{2}|-\rangle\right) = \cos^2\tfrac{\theta}{2} + \sin^2\tfrac{\theta}{2} = 1,$$

$$\langle\pm\theta|\mp\theta\rangle = \left(\cos\tfrac{\theta}{2}\langle+| \pm \sin\tfrac{\theta}{2}\langle-|\right)\left(\cos\tfrac{\theta}{2}|+\rangle \mp \sin\tfrac{\theta}{2}|-\rangle\right) = \cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2} = \cos\theta.$$

We therefore have

$$\langle\psi|\bar{S}_z|\psi\rangle = \langle\theta,\theta\cdots\theta|\frac{\hbar}{2N}\left(|-\theta,\theta\cdots\theta\rangle + |\theta,-\theta\cdots\theta\rangle + \cdots + |\theta,\theta\cdots-\theta\rangle\right)$$

$$= \frac{\hbar}{2N}\left(\cos\theta\cdot 1\cdots 1 + 1\cdot\cos\theta\cdots 1 + \cdots + 1\cdot 1\cdots\cos\theta\right) = \frac{\hbar N}{2N}\cos\theta = \frac{\hbar\cos\theta}{2}.$$

We also need

$$\langle\psi|\bar{S}_z^2|\psi\rangle = \left(\langle-\theta,\theta\cdots\theta| + \langle\theta,-\theta\cdots\theta| + \cdots + \langle\theta,\theta\cdots-\theta|\right)\left(\frac{\hbar}{2N}\right)^2$$

$$\left(|-\theta,\theta\cdots\theta\rangle + |\theta,-\theta\cdots\theta\rangle + \cdots + |\theta,\theta\cdots-\theta\rangle\right).$$

If you multiply this out, there will be N^2 terms in total. For N of them, the "odd" one will be in the same place, for which we have, for example

$$\langle-\theta,\theta\cdots\theta|-\theta,\theta\cdots\theta\rangle = 1\cdot 1\cdots 1 = 1.$$

For the remaining $N^2 - N$ of them, we have

$$\langle-\theta,\theta\cdots\theta|\theta,-\theta\cdots\theta\rangle = \cos\theta\cdot\cos\theta\cdots 1 = \cos^2\theta.$$

Adding this all together, we have

$$\langle\psi|\bar{S}_z^2|\psi\rangle = \left(\frac{\hbar}{2N}\right)^2\left[N\cdot 1 + \left(N^2 - N\right)\cos^2\theta\right] = \frac{\hbar^2}{4}\left[\cos^2\theta + \frac{1}{N}\left(1-\cos^2\theta\right)\right]$$

$$= \frac{\hbar^2}{4}\left(\cos^2\theta + \frac{\sin^2\theta}{N}\right)$$

(d) [4] Show that $\left(\Delta\bar{S}_z\right)^2 = \hbar^2\sin^2\theta/4N$ and show that therefore, in the limit $N \to \infty$, the wave function becomes an eigenstate of $\bar{S}_z$ with the eigenvalue found in part (a).

The uncertainty in $\bar{S}_z$ is

$$\left(\Delta\bar{S}_z\right)^2 = \langle\psi|\bar{S}_z^2|\psi\rangle - \langle\psi|\bar{S}_z|\psi\rangle^2 = \frac{\hbar^2}{4}\left(\cos^2\theta + \frac{\sin^2\theta}{N}\right) - \left[\frac{\hbar\cos\theta}{2}\right]^2 = \frac{\hbar^2\sin^2\theta}{4N}.$$

In the limit $N \to \infty$, this vanishes, so we are in an eigenstate, and note that the expectation value we found in part (c) matches the value in part (a).