Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 11

5. **[20] A general Hermitian operator in a two-dimensional system, such as the state vector for the spin of a spin-1/2 particle, takes the form $\rho = \frac{1}{2}\left(a\mathbf{1} + \mathbf{r}\cdot\boldsymbol{\sigma}\right)$, where $\boldsymbol{\sigma}$ are the Pauli matrices,** 1 **is the unit matrix, and** r **is an arbitrary three-dimensional vector.**
   (a) [4] Find the eigenvalues of this matrix in general.

   Writing $\mathbf{r} = (x, y, z)$, we see that

$$\rho = \tfrac{1}{2}\left(a\mathbf{1} + x\sigma_x + y\sigma_y + z\sigma_z\right) = \frac{1}{2}\begin{pmatrix} a+z & x-iy \\ x+iy & a-z \end{pmatrix}$$

Ignoring the overall factor of ½, we can find the eigenvalues of the remaining matrix λ by demanding

$$0 = \det\begin{pmatrix} a+z-\lambda & x-iy \\ x+iy & a-z-\lambda \end{pmatrix} = (a-\lambda)^2 - z^2 - (x-iy)(x+iy) = (\lambda-a)^2 - x^2 - y^2 - z^2,$$

$$\lambda - a = \pm\sqrt{x^2+y^2+z^2} = \pm|\mathbf{r}|$$

Putting back in the factor of two, we have $\lambda = \frac{1}{2}\left(a \pm |\mathbf{r}|\right)$.

(b) [4] What restrictions can be placed on *a* and r **if this represents a state operator?**

State operators have two restrictions on their eigenvalues: they must have eigenvalues that add to one, and they must be positive. In other words, we must have

$$1 = \tfrac{1}{2}\left(a - |\mathbf{r}|\right) + \tfrac{1}{2}\left(a + |\mathbf{r}|\right) = a,$$
$$0 \le \tfrac{1}{2}\left(a \pm |\mathbf{r}|\right).$$

The first restriction implies $a = 1$. For the second, we have two constraints, but only the minus one yields any information, for which we see that $0 \le a - |\mathbf{r}|$, which implies $|\mathbf{r}| \le 1$. So $a = 1$ and $|\mathbf{r}| \le 1$.

(c) [3] Under what constraints will this density matrix be a pure state?

A pure state has eigenvalues 0 and 1 only, so we must have

$$\lambda = \tfrac{1}{2}\left(a \pm |\mathbf{r}|\right) = \tfrac{1}{2} \pm \tfrac{1}{2}|\mathbf{r}| = 0 \,\text{or}\, 1$$

Obviously, this will happen if $|\mathbf{r}| = 1$.

(d) [4] Show that all four components of *a* and r are determined if we know every component of the expectation value of the spin $\langle \mathbf{S} \rangle$.

We already automatically know that $a = 1$. As for the spin expectation values,

$$\langle S_i \rangle = \tfrac{1}{2}\hbar \mathrm{Tr}(\rho\sigma_i) = \tfrac{1}{4}\hbar \mathrm{Tr}\left[(1+\mathbf{r}\cdot\boldsymbol{\sigma})\sigma_i\right] = \tfrac{1}{4}\hbar \mathrm{Tr}\left(\sigma_i + \sum\nolimits_j r_j\sigma_j\sigma_i\right) = \tfrac{1}{4}\hbar \mathrm{Tr}\left[\sigma_i + \sum\nolimits_j r_j\sigma_j\sigma_i\right]$$
$$= \tfrac{1}{4}\hbar\left[\mathrm{Tr}(\sigma_i) + \sum\nolimits_j r_j \mathrm{Tr}\left(\mathbf{1}\delta_{ij} + \sum\nolimits_k i\varepsilon_{jik}\sigma_k\right)\right] = \tfrac{1}{4}\hbar\left[0 + \sum\nolimits_j r_j\left(\mathrm{Tr}(\mathbf{1})\delta_{ij} + \sum\nolimits_k i\varepsilon_{jik}\mathrm{Tr}(\sigma_k)\right)\right]$$
$$= \tfrac{1}{4}\hbar\left[\sum\nolimits_j r_j\delta_{ij}2 + 0\right] = \tfrac{1}{2}\hbar r_i .$$

This can easily be summarized as $\langle \mathbf{S} \rangle = \tfrac{1}{2}\hbar\mathbf{r}$, so we can get all three components of $\mathbf{r}$ from $\langle \mathbf{S} \rangle$.

(e) [5] A particle with this density matrix is under the influence of a Hamiltonian $H = \tfrac{1}{2}\hbar\omega\sigma_z$. Find a formula for $d\mathbf{r}/dt$ and da/dt, technically four equations, one of which will be trivial.

The state operator (or density matrix) evolves according to

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho],$$
$$\frac{1}{2}\frac{d}{dt}\begin{pmatrix} a+z & x-iy \\ x+iy & a-z \end{pmatrix} = \frac{1}{i\hbar}\frac{\hbar\omega}{2}\frac{1}{2}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a+z & x-iy \\ x+iy & a-z \end{pmatrix} - \begin{pmatrix} a+z & x-iy \\ x+iy & a-z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\},$$
$$\frac{d}{dt}\begin{pmatrix} a+z & x-iy \\ x+iy & a-z \end{pmatrix} = \frac{\omega}{2i}\left\{\begin{pmatrix} a+z & x-iy \\ -x-iy & -a+z \end{pmatrix} - \begin{pmatrix} a+z & -x+iy \\ x+iy & -a+z \end{pmatrix}\right\} = \omega\begin{pmatrix} 0 & -ix-y \\ ix-y & 0 \end{pmatrix}.$$

Equating component by component, we get four simultaneous equations:

$$\frac{d}{dt}a + \frac{d}{dt}z = 0, \quad \frac{d}{dt}a - \frac{d}{dt}z = 0, \quad \frac{d}{dt}x - i\frac{d}{dt}y = -i\omega x - \omega y, \quad \frac{d}{dt}x + i\frac{d}{dt}y = i\omega x - \omega y .$$

We now need to solve these equations for each of the four time derivatives. If we add and subtract the first two, we quickly determine that a and z are unchanging over time. If we add and subtract the last two, we geta $2\,dx/dt = -2\omega y$ and $2i\,dy/dt = 2i\omega x$. Fortunately, these turn into real equations, and our final answer is

$$\frac{d}{dt}a = \frac{d}{dt}z = 0, \quad \frac{d}{dt}x = -\omega y \quad \text{and} \quad \frac{d}{dt}y = \omega x$$

This can be more easily summarized as $\frac{d}{dt}a = 0$ and $\frac{d}{dt}\mathbf{r} = -\omega\hat{\mathbf{z}}\times\mathbf{r}$ if we prefer.