Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 10

6. [15] Two spin-1/2 non-interacting fermions are in the $n = 0$ and $n = 1$ state of a 1D harmonic oscillator; $H = \sum_{i=1}^{2}\left(P_i^2/2m + \frac{1}{2}m\omega^2 X_i^2\right)$. Naively, we might write the quantum state as $|01\rangle$, but this is wrong because (i) it's not antisymmetrized, and (ii) we didn't include spin states

(a) [5] Taking into **account spin states, write four correctly antisymmetrized wave functions, written so they are eigenstates of $\mathbf{S}^2$ and S_z, where S is the total spin of the two electrons. You can write your states in whatever basis is most convenient; I recommend $|n_1 n_2; s m_s\rangle = |n_1 n_2\rangle \otimes |s m_s\rangle$ and/or $|n_1 n_2; m_1 m_2\rangle = |n_1 n_2\rangle \otimes |\pm\pm\rangle$**

The spin states are going to be one of the four states given in (8.21). Note that spin 1 is symmetric, while spin 0 is anti-symmetric, so we can compensate to make the whole state anti-symmetric by making the rest symmetric or anti-symmetric. Our wave state vectors will then be one of the four choices

$$\begin{aligned}
\tfrac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)\otimes|1,+1\rangle &= \tfrac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)\otimes|++\rangle,\\
\tfrac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)\otimes|1,0\rangle &= \tfrac{1}{2}\left(|10\rangle - |01\rangle\right)\otimes\left(|+-\rangle + |-+\rangle\right),\\
\tfrac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)\otimes|1,-1\rangle &= \tfrac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)\otimes|--\rangle,\\
\tfrac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right)\otimes|0,0\rangle &= \tfrac{1}{2}\left(|10\rangle + |01\rangle\right)\otimes\left(|+-\rangle - |-+\rangle\right),
\end{aligned}$$

The spin part plays no role in the Hamiltonian, so it simply factors out.

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(b)[8] For each of the four states, calculate $\langle\psi|(X_1 - X_2)^2|\psi\rangle$

The spin part plays no role. This quantity is most easily calculated as

$$\langle\psi|(X_1 - X_2)^2|\psi\rangle = \left(\langle\psi_x|\otimes\langle\psi_s|\right)(X_1 - X_2)^2\left(|\psi_x\rangle\otimes|\psi_s\rangle\right) = \langle\psi_x|(X_1 - X_2)^2|\psi_x\rangle$$

where $|\psi_x\rangle$ and $|\psi_s\rangle$ are the position and spin parts of the wave function respectively. Hence, we only need to work this out for the position part. We can then rewrite this and use raising and lowering operators to see that

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\begin{aligned}
\langle\psi|(X_1-X_2)^2|\psi\rangle &= \left|(X_1-X_2)|\psi_x\rangle\right|^2 = \left\|\sqrt{\hbar/(2m\omega)}\left(a_1+a_1^\dagger-a_2-a_2^\dagger\right)\tfrac{1}{\sqrt{2}}\left(|10\rangle\mp|01\rangle\right)\right\|^2 \\
&= \frac{\hbar}{4m\omega}\left\|\left(|00\rangle+\sqrt{2}\,|20\rangle-0-|11\rangle\right)\mp\left(0+|11\rangle-|00\rangle-\sqrt{2}\,|02\rangle\right)\right\|^2 \\
&= \frac{\hbar}{4m\omega}\left\|(1\pm1)|00\rangle+\sqrt{2}\,|20\rangle\pm\sqrt{2}\,|02\rangle-(1\pm1)|11\rangle\right\|^2 \\
&= \frac{\hbar}{4m\omega}\left[2+2+(1\pm1)^2+(1\pm1)^2\right]=\begin{cases}3\hbar/m\omega & \text{for spin 1}\\ \hbar/m\omega & \text{for spin 0}\end{cases}
\end{aligned}
$$

(c) [2] Suppose the particles had an additional repulsive force between them, so the fermions prefer to be far apart. Which of the two states would have lower energy?

The particles prefer to be far apart, because they repel. As just calculated, the spin 1 states are farther apart, so spin 1 has lower energy than spin 0, even though spin does not appear in the Hamiltonian.