Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 10

5. [25] A white dwarf is a spherical star consisting of a mix of ^{12}C atoms (6 electrons, atomic mass 12 g/mol) and ^{16}O atoms (8 electrons, atomic mass 16 g/mol). It is under such incredible pressure that the electrons are essentially a free Fermi gas. Similarly, a neutron star consists of almost pure neutrons (atomic mass 1 g/mol). We will approximate each of these objects as uniform spheres of mass *M* and radius *R*. They are held together by gravity, which is opposed by degeneracy pressure. Do all calculations order of magnitude only, so any factors of 2, π, etc., can be treated as one

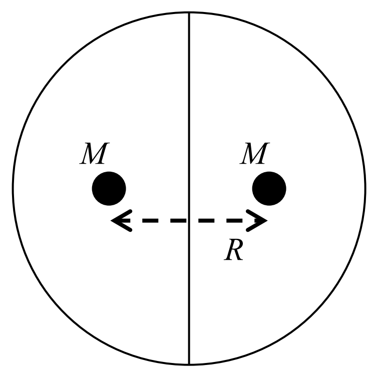
(a) [5] Write a formula applicable in either case for the degeneracy pressure P_F in terms of the mass *M*, the mass of the particle causing the degeneracy pressure *m*, the mass per particle causing degeneracy pressure μ, and the radius *R*. (Note: for the neutron star, μ and *m* are the same thing, but for the white dwarf, they are *very* different, because most of the mass is not in the form of electrons).

If the mass per particle is μ, and the total mass is *M*, then the ratio of these is the number or particles *N*. The number density is number over volume, so

$$n = \frac{N}{V} = \frac{3M}{4\pi\mu R^3} = \frac{M}{\mu R^3}$$

Plugging this into the formula for degeneracy pressure, we have

$$P_F = \frac{\hbar^2}{5m}\left(3\pi^2\right)^{2/3}\left(\frac{N}{V}\right)^{5/3} = \frac{\hbar^2}{m}\left(\frac{M}{\mu R^3}\right)^{5/3} = \frac{\hbar^2}{mR^5}\left(\frac{M}{\mu}\right)^{5/3}$$



(b) [5] Multiply the degeneracy pressure by the cross-sectional area of the star sliced in half. Equate this to the gravitational attraction, which you estimate as follows: pretend each half of the star has mass *M*/2, and are separated by a distance *R*. Find an equation for *R* in terms of *M* (and other constants).

The cross-sectional area is just $A = \pi R^2 = R^2$, so the total degeneracy force is

$$F_F = \frac{\hbar^2}{mR^3}\left(\frac{M}{\mu}\right)^{5/3}.$$

We now set this equal to the gravitational force,

$$F_G = \frac{GM^2}{R^2} = \frac{\hbar^2}{mR^3}\left(\frac{M}{\mu}\right)^{5/3},$$

$$R = \frac{\hbar^2}{GmM^2}\left(\frac{M}{\mu}\right)^{5/3} = \frac{\hbar^2}{GmM^{1/3}\mu^{5/3}}.$$

One interesting fact is that the radius shrinks as the mass increases!

(c) [5] What is μ for a ^{12}C or ^{16}O white dwarf? Find the numerical value of (b) in km for a white dwarf or neutron star of mass equal to the Sun (M = 1.989 × 10^{30} kg).

The mass per electron in each of these cases is

$$\mu_C = \frac{(12.0000\text{ g/mol})(10^{-3}\text{ kg/g})}{6(6.022\times10^{23}\text{ /mol})} = 3.321\times10^{-27}\text{ kg}$$

$$\mu_O = \frac{(15.9949\text{ g/mol})(10^{-3}\text{ kg/g})}{8(6.022\times10^{23}\text{ /mol})} = 3.320\times10^{-27}\text{ kg}$$

Unless we want more than four digit accuracy, these are identical, so there's no point in doing them separately. We now just crunch our numbers.

$$R_{wd} = \frac{(1.055\times10^{-34}\text{ J}\cdot\text{s})^2}{(6.6742\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1.989\times10^{30}\text{ kg})^{1/3}(9.109\times10^{-31}\text{ kg})(3.321\times10^{-27}\text{ kg})^{5/3}}$$
$$= 1.97\times10^6\text{ m} = 1{,}970\text{ km},$$

$$R_{ns} = \frac{(1.055\times10^{-34}\text{ J}\cdot\text{s})^2}{(6.6742\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1.989\times10^{30}\text{ kg})^{1/3}(1.675\times10^{-27}\text{ kg})^{8/3}} = 3{,}350\text{ m} = 3.35\text{ km}.$$

The white dwarf is too small by about a factor of 3, and the neutron star by about a factor of five.

(d) [4] Find a formula for the Fermi energy E_F in terms of the mass M (and other constants).

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{\frac{2}{3}} = \frac{\hbar^2}{m}n^{\frac{2}{3}} = \frac{\hbar^2}{m}\left(\frac{M}{\mu R^3}\right)^{\frac{2}{3}} = \frac{\hbar^2}{m}\left(\frac{M}{\mu}\frac{G^3 m^3 M\mu^5}{\hbar^6}\right)^{\frac{2}{3}} = \frac{G^2 m}{\hbar^2}(M\mu^2)^{4/3}$$

(e) [6] When the Fermi energy reaches mc^2, the approximations break down, and the star undergoes catastrophic collapse. Estimate this mass *M* for each of the two cases, in solar masses, where this collapse occurs. The actual values are 1.4 solar masses (white dwarf) and around 2.5 solar masses (neutron star).

We want to equate this to mc^2 and solve for *M*. We find

$$\begin{aligned} mc^2 &= \frac{G^2 m}{\hbar^2}\left(M\mu^2\right)^{4/3} \\ \left(M\mu^2\right)^{4/3} &= \frac{\hbar^2 c^2}{G^2}, \\ M\mu^2 &= \left(\frac{\hbar c}{G}\right)^{3/2}, \\ M &= \frac{1}{\mu^2}\left(\frac{\hbar c}{G}\right)^{3/2}. \end{aligned}$$

It's interesting that the mass of the particle *m* disappears entirely from the computation.

$$M_{wd} = \frac{1}{\left(3.321\times10^{-27}\ \text{kg}\right)^2}\left[\frac{\left(1.055\times10^{-34}\ \text{J}\cdot\text{s}\right)\left(2.998\times10^{8}\ \text{m/s}\right)}{6.6742\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2}\right]^{3/2} = 9.353\times10^{29}\ \text{kg} = 0.470 M_\odot,$$

$$M_{ns} = \frac{1}{\left(1.675\times10^{-27}\ \text{kg}\right)^2}\left[\frac{\left(1.055\times10^{-34}\ \text{J}\cdot\text{s}\right)\left(2.998\times10^{8}\ \text{m/s}\right)}{6.6742\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2}\right]^{3/2} = 3.670\times10^{30}\ \text{kg} = 1.85 M_\odot.$$

The white dwarf estimate is clearly too small, but it is in the right ballpark. The neutron star estimate is also a little small.