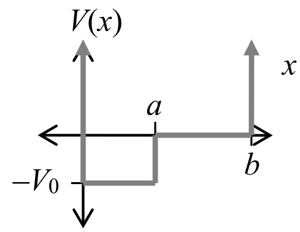
Quantum Mechanics Graduate Exam

Summer, 2021

Each problem is worth 25 points. The points for individual parts are marked in square brackets. **To ensure full credit, show your work.** Do any four (4) of the following five (5) problems. If you attempt all 5 problems you must clearly state which 4 problems you want to have graded.

1. A particle of mass m has energy eigenvalue $E = 0$ in the 1D potential sketched below,

$$V(x) = \begin{cases} -V_0 & \text{if } 0 < x < a, \\ 0 & \text{if } a < x < b, \\ \infty & \text{otherwise.} \end{cases}$$



Find an expression for b in terms of a, V_0 and m.

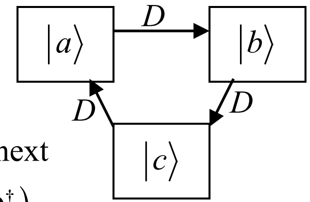
2. An electron is in a magnetic field pointing in the z-direction, $\mathbf{B} = B\hat{\mathbf{z}}$. The Hamiltonian is given by $H = -\gamma \mathbf{B} \cdot \mathbf{S}$. Several helpful formulas appear at the end of this problem.
   (a) [2] What are the eigenvalues and eigenvectors for the energy?
   (b) [6] If the system at $t = 0$ is in the state $|\Psi(0)\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, what is the state at time t, $|\Psi(t)\rangle$?
   (c) [6] Calculate the expectation value of the three spin operators $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$ at time t.
   (d) [6] If the spin along the x-direction is measured at time t what is the probability that it will have the value $+\frac{1}{2}\hbar$?
   (e) [5] The measurement is performed, and the result does come out to be $s_x = +\frac{1}{2}\hbar$. What is the state vector at time t' after the measurement, up to an irrelevant phase?

Helpful formulas: $S_i = \frac{1}{2}\hbar\sigma_i$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

3. A particle of mass m in the Harmonic oscillator with frequency ω is in the state $|\psi\rangle = \frac{1}{2}\left(|0\rangle + |1\rangle + \sqrt{2}|2\rangle\right)$.
   (a) [4] What is the expectation value of the Hamiltonian $\langle H \rangle$?
   (b) [16] What are the expectation values of $\langle X \rangle$, $\langle X^2 \rangle$, $\langle P \rangle$, and $\langle P^2 \rangle$?
   (c) [5] Find the uncertainties Δx and Δp, and check that it satisfies the uncertainty relation.

<u>Harmonic Oscillator Formulas</u>

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad X = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right), \quad P = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right).$$



4. A quantum mechanical system consists of three orthonormal basis vectors $\{|a\rangle,|b\rangle,|c\rangle\}$, and the operator D is defined by $D|a\rangle=|b\rangle$, $D|b\rangle=|c\rangle$, and $D|c\rangle=|a\rangle$, so that *D* causes each state to "hop" to the next state. The Hamiltonian in this basis is given by $H=-\hbar\omega\left(D^\dagger D+D+D^\dagger\right)$.
   (a) [6] Write the matrix representation of *D*, $D^\dagger$ and *H* in the $\{|a\rangle,|b\rangle,|c\rangle\}$ basis.
   (b) [2] Can the operator *D* correspond to a physical observable? Justify your answer (1 sentence).
   (c) [6] Show that the states $|\psi_1\rangle=\frac{1}{\sqrt{3}}\left(|a\rangle+|b\rangle+|c\rangle\right)$, $|\psi_2\rangle=\frac{1}{\sqrt{2}}\left(|a\rangle-|b\rangle\right)$, and $|\psi_3\rangle=\frac{1}{\sqrt{6}}\left(|a\rangle+|b\rangle-2|c\rangle\right)$ are eigenstates of the Hamiltonian, and find their eigenvalues.
   (d) [11] If the system starts in state $|\Psi(0)\rangle=|c\rangle$ at time 0, what is the state at an arbitrary time $|\Psi(t)\rangle$? If the state is measured at time *t*, what is the probability it will be in the state $|c\rangle$?

5. Consider a quantum system with just *three* linearly independent states. The Hamiltonian, in matrix form, is

$$H=V_0\begin{pmatrix}(1-\varepsilon) & 0 & 0\\ 0 & 1 & \varepsilon\\ 0 & \varepsilon & 2\end{pmatrix}$$

   Where V_0 is constant and ε is some small parameter ($\varepsilon \ll 1$).
   (a) [2] Write down the eigenvectors and eigenvalues of the *unperturbed* Hamiltonian H_0 where $\varepsilon=0$.
   (b) [5] Use first- and second-order *non*-degenerate perturbation theory to find the approximate eigenvalue for the state that comes from the non-degenerate eigenvector of H_0.
   (c) [7] Use *degenerate* perturbation theory to find the first-order correction to the two initially degenerate eigenvalues.
   (d) [11] Solve for the **exact** eigenvalues of *H*. Expand each of them as a power series in ε up to second order. Compare with the results of perturbation theory.