Physics 744 - Field Theory

Solution Set 7

1. In the $\psi^*\psi\phi$ theory, consider the tree-level diagrams (no loops) contributing to the scattering $\psi(\mathbf{p}_1)\psi^*(\mathbf{p}_2)\to\psi(\mathbf{p}_1')\psi^*(\mathbf{p}_2')$.

(a) Draw all (two) Feynman diagrams that contribute to this process and label the intermediate momenta. Write the corresponding Feynman amplitude.

The two diagrams are sketched above. The Feynman invariant amplitude will be given by

$$iM = (-i\gamma)^2\left\{\frac{i}{(\mathbf{p}_1-\mathbf{p}_1')^2-M^2}+\frac{i}{(\mathbf{p}_1+\mathbf{p}_2)^2-M^2}\right\}.$$

We have neglected the $i\varepsilon$ contribution in the denominator. This will be important if we are right on the energy of the intermediate particle, as can happen in the second term, but in that case we need to be more sophisticated about everything anyway.

(b) Work out the differential cross-section in the center of mass frame. You may write your answer in terms of the energies *E* of any one of the particles or the magnitude of the three-momentum *p* as needed. Let θ represent the angle between the initial and final momenta of the ψ's.

The square of any particle's mass is just m^2. Since we are in the center of mass frame, the incoming particles have equal and opposite momenta. Since they have the same mass, that means they have the same energy as well. By conservation of momentum, the final particles must also have equal and opposite momenta, and since their masses are identical, their energies will be as well. Hence all four particles have the same energy E and the same magnitude of their moment p, with $E^2 = p^2+m^2$. The dot products are not too difficult to work out:

$$\mathbf{p}_1\cdot\mathbf{p}_1' = E^2-\vec{p}_1\cdot\vec{p}_1' = E^2-p^2\cos\theta,$$
$$\mathbf{p}_1\cdot\mathbf{p}_2 = E^2-\vec{p}_1\cdot\vec{p}_2 = E^2+p^2.$$

We therefore have

$$i\mathcal{M} = -\gamma^2 \left\{ \frac{1}{\mathbf{p}_1^2 + \mathbf{p}_1'^2 - 2\mathbf{p}_1 \cdot \mathbf{p}_1' - M^2} + \frac{1}{\mathbf{p}_1^2 + \mathbf{p}_2^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 - M^2} \right\}$$

$$= -\gamma^2 \left\{ \frac{1}{2m^2 - 2E^2 + 2p^2 \cos^2\theta - M^2} + \frac{1}{2m^2 + 2E^2 + 2p^2 - M^2} \right\}$$

$$= \gamma^2 \left[\frac{1}{M^2 - 4E^2} + \frac{1}{M^2 + 2p^2(1 - \cos\theta)} \right].$$

The cross-section is then given by

$$\sigma |\Delta\vec{v}| = \frac{1}{4EE'} \int \frac{p\,d\Omega}{16\pi^2 E_{\text{tot}}} |i\mathcal{M}|^2 ,$$

$$\sigma \frac{2p}{E} = \frac{p\gamma^4}{4E^2 16\pi^2 (2E)} \int \left[\frac{1}{M^2 - 4E^2} + \frac{1}{M^2 + 2p^2(1 - \cos\theta)} \right]^2 d\Omega,$$

$$\frac{d\sigma}{d\Omega} = \frac{\gamma^4}{256\pi^2 E^2} \left[\frac{1}{M^2 - 4E^2} + \frac{1}{M^2 + 2p^2(1 - \cos\theta)} \right]^2$$

where we simply declined to do the angular integral in the final step.

(c) Are there any subtleties having to do with final momenta? Find the total cross-section.

The final state particles are non-identical, so in this case there are no subtleties. We simply have to do the final integral. The ϕ integral is trivial, and we let $x = 1 - \cos\theta$ to obtain

$$\sigma = \frac{\gamma^4}{128\pi E^2} \int_0^2 \left[\frac{1}{(M^2 - 4E^2)^2} + \frac{2}{(M^2 - 4E^2)(M^2 + 2p^2 x)} + \frac{1}{(M^2 + 2p^2 x)^2} \right] dx$$

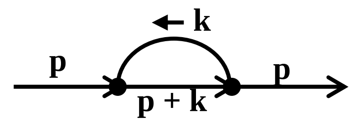
$$= \frac{\gamma^4}{64\pi E^2} \left[\frac{1}{(M^2 - 4E^2)^2} + \frac{\ln(M^2 + 2p^2 x)\big|_0^2}{2(M^2 - 4E^2)p^2} - \frac{1}{4p^2(M^2 + 2p^2 x)}\Bigg|_0^2 \right]$$

$$= \frac{\gamma^4}{32\pi E^2} \left[\frac{1}{(M^2 - 4E^2)^2} + \frac{\ln(1 + 4p^2/M^2)}{2(M^2 - 4E^2)p^2} + \frac{1}{M^2(M^2 + 4p^2)} \right]$$

It isn’t pretty, but we’re done.

2. We wish to work out the one loop contribution to the propagator $\pi\left(p^2\right)$ for the ψ particle in the $\psi^*\psi\phi$ theory, using dimensional regularization.

(a) Draw the relevant one loop diagram and write an expression for the Feynman amplitude.



The relevant Feynman diagram is sketched at right. The Feynman amplitude is

$$-i\pi\left(p^2\right)=\left(-i\gamma\right)^2\int\frac{d^4\mathbf{k}}{\left(2\pi\right)^4}\frac{i}{\mathbf{k}^2-M^2+i\varepsilon}\frac{i}{\left(\mathbf{k}+\mathbf{p}\right)^2-m^2+i\varepsilon}$$

(b) Combine the denominators using Feynman parameters. Shift the integral to make it spherically symmetric.

Following the instructions, we have

$$\begin{aligned}-i\pi\left(p^2\right)&=\gamma^2\int_0^1 dx\int\frac{d^4\mathbf{k}}{\left(2\pi\right)^4}\frac{1}{\left\{\left[\left(\mathbf{k}+\mathbf{p}\right)^2-m^2\right]x+\left(1-x\right)\left(\mathbf{k}^2-M^2\right)+i\varepsilon\right\}^2}\\
&=\gamma^2\int_0^1 dx\int\frac{d^4\mathbf{k}}{\left(2\pi\right)^4}\frac{1}{\left[\mathbf{k}^2+2x\mathbf{k}\cdot\mathbf{p}+x\mathbf{p}^2-xm^2-\left(1-x\right)M^2+i\varepsilon\right]^2}\\
&=\gamma^2\int_0^1 dx\int\frac{d^4\mathbf{k}}{\left(2\pi\right)^4}\frac{1}{\left[\left(\mathbf{k}+x\mathbf{p}\right)^2+\left(x-x^2\right)\mathbf{p}^2-xm^2-\left(1-x\right)M^2+i\varepsilon\right]^2}\\
&=\gamma^2\int_0^1 dx\int\frac{d^4\mathbf{k}}{\left(2\pi\right)^4}\frac{1}{\left[\mathbf{k}^2+\left(x-x^2\right)\mathbf{p}^2-xm^2-\left(1-x\right)M^2+i\varepsilon\right]^2}\end{aligned}$$

(c) Regulate the integral using dimensional regularization in $d=4-2\varepsilon$ dimensions. Perform the momentum integrals.

We now switch to $d=4-2\varepsilon$ dimensions, and immediately perform the momentum integrals, using our formula. We then have

$$\begin{aligned}-i\pi\left(p^2\right)&=\gamma^2\int_0^1 dx\int\frac{d^d\mathbf{k}}{\left(2\pi\right)^d}\frac{1}{\left[\mathbf{k}^2+\left(x-x^2\right)\mathbf{p}^2-xm^2-\left(1-x\right)M^2+i\varepsilon\right]^2}\\
&=\gamma^2\int_0^1 dx\frac{i\left(-\right)^2}{\left(4\pi\right)^{d/2}\left[xm^2+\left(1-x\right)M^2-\left(x-x^2\right)\mathbf{p}^2-i\varepsilon\right]^{2-d/2}}\frac{\Gamma\left(2-d/2\right)}{\Gamma\left(2\right)}\frac{\Gamma\left(d/2\right)}{\Gamma\left(d/2\right)}\end{aligned}$$

$$\pi\left(p^2\right)=-\frac{\gamma^2}{16\pi^2}\int_0^1 dx\Gamma\left(\varepsilon\right)\left[\frac{xm^2+\left(1-x\right)M^2-\left(x-x^2\right)\mathbf{p}^2-i\varepsilon}{4\pi}\right]^{-\varepsilon}$$

(d) Multiply out all the factors, keeping terms of $\mathcal{O}\left(\varepsilon^{-1}\right)$ and $\mathcal{O}\left(1\right)$, but dropping lower terms. You may leave one Feynman parameter undone.

The small power can be approximated by $a^{\varepsilon}\approx\exp\left(\varepsilon\ln a\right)=1+\varepsilon\ln a$, and $\Gamma\left(\varepsilon\right)=\varepsilon^{-1}-\gamma$. Note that this γ is Euler's constant, having nothing to do with the coupling γ. We therefore have

$$\begin{aligned}\pi\left(p^2\right)&=\frac{\gamma^2}{16\pi^2}\int_0^1 dx\left(-\frac{1}{\varepsilon}+\gamma\right)\left\{1-\varepsilon\ln\left[xm^2+\left(1-x\right)M^2-\left(x-x^2\right)\mathbf{p}^2-i\varepsilon\right]-\varepsilon\ln\left(4\pi\right)\right\}\\&=\frac{\gamma^2}{16\pi^2}\int_0^1 dx\left\{-\frac{1}{\varepsilon}+\gamma+\ln\left[xm^2+\left(1-x\right)M^2-\left(x-x^2\right)p^2-i\varepsilon\right]-\ln\left(4\pi\right)\right\}\end{aligned}$$

The integrals are all doable, but we won't go to the trouble of actually doing them.

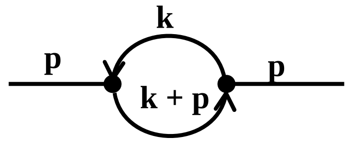
(e) Convince yourself, and me, that $\pi\left(m^2\right)$ is always real. Hence there is no problem calculating the counterterm.

If you substitute $p^2=m^2$, this simplifies to

$$\pi\left(m^2\right)=\frac{g^2}{16\pi^2}\int_0^1 dx\left\{-\frac{1}{\varepsilon}+\gamma+\ln\left[x^2m^2+\left(1-x\right)M^2-i\varepsilon\right]-\ln\left(4\pi\right)\right\}$$

Now, each of the terms in the logarithm is positive, and therefore we can take the limit $\varepsilon\to 0$ with impunity. We can then add appropriate counterterms to make sure everything vanishes at $p^2 = m^2$.

3. We wish to work out the one loop contribution to the propagator $\pi\left(p^2\right)$ for the ϕ particle in the $\psi^*\psi\phi$ theory, using dimensional regularization. This was done in class, but I want you to redo it using dimensional regularization. (a-d) Same as previous problem.



The appropriate Feynman diagram is given above. We now calculate

$$
\begin{aligned}
-i\pi\left(p^2\right) &= \left(-i\gamma\right)^2 \int \frac{d^4\mathbf{k}}{\left(2\pi\right)^4} \frac{i}{\mathbf{k}^2 - m^2 + i\varepsilon} \frac{i}{\left(\mathbf{k}+\mathbf{p}\right)^2 - m^2 + i\varepsilon} \\
&= \gamma^2 \int_0^1 dx \int \frac{d^4\mathbf{k}}{\left(2\pi\right)^4} \frac{1}{\left\{\left(\mathbf{k}^2 - m^2 + i\varepsilon\right)\left(1-x\right) + \left[\left(\mathbf{k}+\mathbf{p}\right)^2 - m^2 + i\varepsilon\right]x\right\}} \\
&= \gamma^2 \int_0^1 dx \int \frac{d^4\mathbf{k}}{\left(2\pi\right)^4} \frac{1}{\left(\mathbf{k}^2 + 2x\mathbf{k}\cdot\mathbf{p} + x\mathbf{p}^2 - m^2 + i\varepsilon\right)^2} \\
&= \gamma^2 \int_0^1 dx \int \frac{d^4\mathbf{k}}{\left(2\pi\right)^4} \frac{1}{\left[\left(\mathbf{k} + x\mathbf{p}\right)^2 + \left(x - x^2\right)\mathbf{p}^2 - m^2 + i\varepsilon\right]^2} \\
&= \gamma^2 \int_0^1 dx \int \frac{d^4\mathbf{k}}{\left(2\pi\right)^4} \frac{1}{\left[\mathbf{k}^2 + \left(x - x^2\right)\mathbf{p}^2 - m^2 + i\varepsilon\right]^2} \\
&\to \gamma^2 \int_0^1 dx \int \frac{d^d\mathbf{k}}{\left(2\pi\right)^d} \frac{1}{\left[\mathbf{k}^2 + \left(x - x^2\right)\mathbf{p}^2 - m^2 + i\varepsilon\right]^2} \\
&= \gamma^2 \int_0^1 dx \frac{i\left(-\right)^2}{\left(4\pi\right)^{d/2}\left[m^2 - \left(x - x^2\right)\mathbf{p}^2 - i\varepsilon\right]^{2-d/2}} \frac{\Gamma\left(2 - d/2\right)}{\Gamma\left(2\right)} \frac{\Gamma\left(d/2\right)}{\Gamma\left(d/2\right)} \\
&= \frac{i\gamma^2}{16\pi^2} \Gamma\left(2 - d/2\right) \left\{\left[m^2 - \left(x - x^2\right)p^2 - i\varepsilon\right] / 4\pi\right\}^{d/2-2} \\
&= \frac{i\gamma^2}{16\pi^2} \left(\frac{1}{\varepsilon} - \gamma\right) \left\{1 - \varepsilon \ln\left[m^2 - \left(x - x^2\right)p^2 - i\varepsilon\right] + \varepsilon \ln\left(4\pi\right)\right\},
\end{aligned}
$$

$$
\pi\left(p^2\right) = \frac{\gamma^2}{16\pi^2} \int_0^1 dx \left\{-\varepsilon^{-1} + \gamma + \ln\left[m^2 - \left(x - x^2\right)p^2 - i\varepsilon\right] - \ln\left(4\pi\right)\right\}.
$$

(e) Convince yourself, and me, that $\pi\left(M^2\right)$ is real if $M < 2m$. If $M > 2m$, find the imaginary part, and compare its value to the decay rate for the ϕ, given by $\Gamma = \gamma^2\sqrt{M^2 - 4m^2}\big/16\pi M^2$.

The problem, if any, must lie with the logarithm. The logarithm is $\ln\left[m^2 - M^2\left(x - x^2\right) - i\varepsilon\right]$. The function $x - x^2$ rises from 0 to a maximum value of $\frac{1}{4}$ at $x = \frac{1}{2}$, so the expression $m^2 - M^2\left(x - x^2\right)$ is never smaller than $m^2 - \frac{1}{4}M^2$, and if $M < 2m$, this is positive. Hence we are simply taking the logarithm of a positive number, and therefore $\pi\left(M^2\right)$ is real. Hence we can add counterterms to make it go away.

If, on the other hand, $M > 2m$, then for some values of x the logarithm will be of a negative number, which will yield a logarithm with an imaginary part. The imaginary part is $-i\pi$ whenever $m^2 - M^2\left(x - x^2\right)$ is negative, so we need to figure out when this happens. Setting it equal to zero, we see that the places where it crosses the x-axis are when

$$m^2 - M^2\left(x - x^2\right) = 0,$$
$$x^2 - x + m^2/M^2 = 0,$$
$$x_\pm = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - m^2/M^2}$$

The range of x-values where the function is negative is therefore

$$\Delta x = x_+ - x_- = 2\sqrt{\tfrac{1}{4} - m^2/M^2} = \sqrt{1 - 4m^2/M^2}\,.$$

The imaginary part of $\pi\left(M^2\right)$ will therefore be

$$\operatorname{Im}\left\{\pi\left(M^2\right)\right\} = \frac{\gamma^2}{16\pi^2}\int_{x_-}^{x_+} dx\,\operatorname{Im}\left\{-i\pi\right\} = -\frac{\gamma^2}{16\pi}\sqrt{1 - 4m^2/M^2} = -\frac{\gamma^2\sqrt{M^2 - 4m^2}}{16\pi M}$$

It is then easy to see that $\operatorname{Im}\left\{\pi\left(M^2\right)\right\} = -M\Gamma$. This is not a coincidence, but we won't go into it now.

Useful formula: $$\lim_{\varepsilon\to 0}\left[\ln\left(x - i\varepsilon\right)\right] = \begin{cases} \ln x & \text{if } x > 0, \\ -i\pi + \ln|x| & \text{if } x < 0. \end{cases}$$