Physics 744 – Quantum Field Theory

Homework Set 1

1. A set of particles in three dimensions ($\vec{r}_a = (x_a, y_a, z_a)$) interacts via the Lagrangian

$$L(\vec{r}_a, \dot{\vec{r}}_a) = \sum_a \tfrac{1}{2} m_a \dot{\vec{r}}_a \cdot \dot{\vec{r}}_a - \sum_a W_a(|\vec{r}_a|) - \sum_{a<b} V_{ab}(|\vec{r}_a - \vec{r}_b|)$$

where W_a and V_{ab} are arbitrary functions of the magnitudes listed. Consider a set of new coordinates

$$x_a' = x_a \cos\theta - y_a \sin\theta$$
$$y_a' = y_a \cos\theta + x_a \sin\theta$$
$$z_a' = z_a$$

(a) Show that $L(\vec{r}_a', \dot{\vec{r}}_a') = L(\vec{r}_a, \dot{\vec{r}}_a)$, and that therefore the derivative of the left-hand side with respect to θ is trivial.

(b) Deduce the corresponding conserved quantity, and identify it.

2. A set of particles in three dimensions ($\vec{r}_a = (x_a, y_a, z_a)$) interacts via the Lagrangian

$$L(\vec{r}_a, \dot{\vec{r}}_a) = \sum_a \tfrac{1}{2} m_a \dot{\vec{r}}_a \cdot \dot{\vec{r}}_a - \sum_{a<b} V_{ab}(\vec{r}_a - \vec{r}_b)$$

where the V_{ab} are arbitrary functions of the differences of the coordinates. Consider the Galilean transformation

$$x_a' = x_a + vt$$
$$y_a' = y_a$$
$$z_a' = z_a$$

(a) Calculate $L(\vec{r}_a', \dot{\vec{r}}_a')$, and show that although its derivative with respect to v at $v = 0$ is non-zero, it can be written as a total time derivative of some quantity.

(b) Find a quantity which is consequently conserved; that is, whose time derivative is zero. Write this quantity in terms of the total mass M, some part of the total momentum $\vec{P} = (P_x, P_y, P_z)$, and the center of mass coordinate $\vec{R} = (X, Y, Z)$.

3. Two particles are moving in one dimension with Lagrangian

$$L = \tfrac{1}{2}M\dot{x}_1^2 + 2M\dot{x}_2^2 - \tfrac{1}{2}M\omega^2\left[x_1^2 + 4x_1x_2 + 10x_2^2\right]$$

(a) Find a change of variables $x_1, x_2 \to y_1, y_2$ so that the Lagrangian, rewritten in terms of the y's, takes the form

$$L = \tfrac{1}{2}M\left(\dot{y}_1^2 + \dot{y}_2^2\right) - \tfrac{1}{2}K_{ij}y_iy_j$$

what is the matrix K? Note that K_{ij} must be symmetric, so the coefficient of y_1y_2 must be cut in half to deduce $K_{12} = K_{21}$.

(b) Find the eigenvalues of K and the corresponding orthonormal eigenvectors. If the eigenvalues of K are complicated, then either you or I have made a mistake.

(c) Find a change of variables $y_1, y_2 \to z_1, z_2$ such that the Lagrangian now takes the form

$$L = \tfrac{1}{2}M\left(\dot{z}_1^2 + \dot{z}_2^2\right) - \tfrac{1}{2}k_1z_1^2 - \tfrac{1}{2}k_2z_2^2$$

(d) What are the normal frequencies of this system?