Solutions to Problems 9b

6. An up quark is scattering off of a down quark, $u(p_1)d(p_2)\to u(p_3)d(p_4)$, due to strong interactions. Treat the quarks as massless, and assume the initial quarks are of random spin and color. Find the differential cross section, as well as the total cross-section assuming there is a minimum angle $\theta_{\min}$ for which the scattering is distinguishable from not scattering.

There is only one diagram, sketched at right. I have included the colors of all the relevant quarks. The Feynman amplitude is

$$i\mathcal{M}=\left(-ig_s\right)^2\left(T_a\right)^i{}_k\left(T_a\right)^j{}_l\left(\bar{u}_3\gamma^\mu u_1\right)\left(\bar{u}_4\gamma^\nu u_2\right)\frac{-ig_{\mu\nu}}{\left(p_1-p_3\right)^2}=-\frac{ig_s^2}{2p_1\cdot p_3}\left(T_a\right)^i{}_k\left(T_a\right)^j{}_l\left(\bar{u}_3\gamma^\mu u_1\right)\left(\bar{u}_4\gamma_\mu u_2\right).$$

Comparing with the work or problem 7.7 we see that this is identical with the amplitude we got for $e^- f\to e^- f$, except that e has been replaced by g_s, and Q has been replaced by $-\left(T_a\right)^i{}_k\left(T_a\right)^j{}_l$. This allows us to jump immediately to the differential cross-section, assuming we know the initial and final colors we are interested in. The result is

$$\left(\frac{d\sigma}{d\Omega}\right)_{ijkl}=\frac{\alpha_s^2}{8E^2}\left|\left(T_a\right)^i{}_k\left(T_a\right)^j{}_l\right|^2\cdot\frac{\cos^2\theta+2\cos\theta+5}{\left(1-\cos\theta\right)^2}.$$

As in the notes, we first rewrite this in the form

$$\left(\frac{d\sigma}{d\Omega}\right)_{ijkl}=\frac{\alpha_s^2}{8E^2}\left(T_a\right)^i{}_k\left(T_a\right)^j{}_l\left(T_b\right)^k{}_i\left(T_b\right)^l{}_j\cdot\frac{\cos^2\theta+2\cos\theta+5}{\left(1-\cos\theta\right)^2}\quad\text{(no sum on } i,j,k,l\text{)}.$$

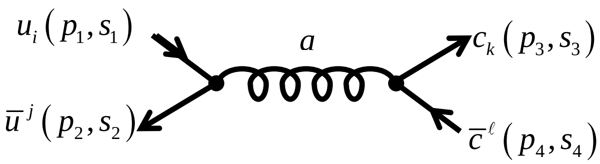
So far so good. But we now average over the initial colors and sum over the final colors, which gives us

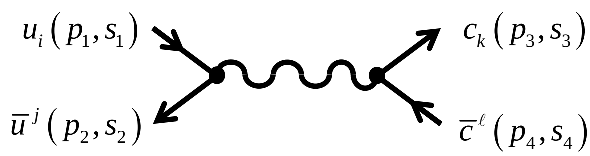
$$\frac{d\sigma}{d\Omega}=\frac{\alpha_s^2}{8E^2}\left(T_a\right)^i{}_k\left(T_a\right)^j{}_l\left(T_b\right)^k{}_i\left(T_b\right)^l{}_j\cdot\frac{\cos^2\theta+2\cos\theta+5}{\left(1-\cos\theta\right)^2}$$
$$=\frac{\alpha_s^2}{72E^2}\mathrm{Tr}\left(T_aT_b\right)\mathrm{Tr}\left(T_aT_b\right)\frac{\cos^2\theta+2\cos\theta+5}{\left(1-\cos\theta\right)^2}=\frac{\alpha_s^2}{288E^2}\delta_{ab}\delta_{ab}\frac{\cos^2\theta+2\cos\theta+5}{\left(1-\cos\theta\right)^2}$$
$$=\frac{\alpha_s^2\left(\cos^2\theta+2\cos\theta+5\right)}{36E^2\left(1-\cos\theta\right)^2}.$$

The integral is identical to problem 7.7, so if we wish we can then get the total cross-section, which is

$$\sigma=\frac{\pi\alpha_S^2}{9E^2}\left[\cot^2\left(\tfrac{1}{2}\theta_{\min}\right)+\tfrac{1}{2}\cos^2\left(\tfrac{1}{2}\theta_{\min}\right)+\ln\left(\sin\left(\tfrac{1}{2}\theta_{\min}\right)\right)\right].$$

8. In eq. (9.27), we found the cross section for $u\bar{u} \to c\bar{c}$, using only the tree-level QCD contribution. Add in the QED contribution to the amplitude. Convince yourself (and me) that the cross terms $\left(i\mathcal{M}_{QCD}\right)\left(i\mathcal{M}_{QED}\right)^*$ vanishes. Find the total cross-section.





There are two diagrams, sketched above. The combined amplitude is

$$
\begin{aligned}
i\mathcal{M} &= \frac{ig_s^2}{s}\left(T_a\right)^j{}_i\left(T_a\right)^k{}_\ell\left(\bar{v}_2\gamma^\mu u_1\right)\left(\bar{u}_3\gamma_\mu v_4\right) + \frac{ie^2}{s}\left(\frac{2}{3}\right)^2\delta^j{}_i\delta^k{}_l\left(\bar{v}_2\gamma^\mu u_1\right)\left(\bar{u}_3\gamma_\mu v_4\right) \\
&= \frac{i}{s}\left(\bar{v}_2\gamma^\mu u_1\right)\left(\bar{u}_3\gamma_\mu v_4\right)\left[g_s^2\left(T_a\right)^j{}_i\left(T_a\right)^k{}_\ell + \tfrac{4}{9}e^2\delta^j{}_i\delta^k{}_l\right].
\end{aligned}
$$

The factors of 2/3 come from the up and charm quark masses respectively. In a manner identical to what we did in the notes, this then yields a cross section *if* we knew all the colors of the initial and final states of

$$
\begin{aligned}
\sigma_{ijk\ell} &= \frac{4\pi\alpha_s^2}{3s}\left[\left(T_a\right)^j{}_i\left(T_a\right)^k{}_\ell + \tfrac{4}{9}\delta^j{}_i\delta^k{}_\ell\right]\left[\left(T_b\right)^{j\,*}_{\ i}\left(T_b\right)^{k\,*}_{\ \ell} + \tfrac{4}{9}\delta^j{}_i\delta^k{}_\ell\right] \\
&= \frac{4\pi\alpha_s^2}{3s}\left[\begin{matrix}\left(T_a\right)^j{}_i\left(T_b\right)^i{}_j\left(T_a\right)^k{}_\ell\left(T_b\right)^\ell{}_k + \tfrac{4}{9}\left(T_a\right)^j{}_i\delta^i{}_j\left(T_a\right)^k{}_\ell\delta^\ell{}_k \\ +\tfrac{4}{9}\left(T_b\right)^i{}_j\left(T_b\right)^\ell{}_k\delta^j{}_i\delta^k{}_\ell + \tfrac{16}{81}\delta^i{}_j\delta^j{}_i\delta^\ell{}_k\delta^k{}_\ell\end{matrix}\right], \qquad \text{(no sum on } i,j,k,\ell\text{)}
\end{aligned}
$$

Now we need to sum over final colors and average over initial colors. This yields

$$
\begin{aligned}
\sigma &= \frac{4\pi}{27s}\left[\begin{matrix}\alpha_s^2\left(T_a\right)^j{}_i\left(T_b\right)^i{}_j\left(T_a\right)^k{}_\ell\left(T_b\right)^\ell{}_k + \tfrac{4}{9}\alpha_s\alpha\left(T_a\right)^j{}_i\delta^i{}_j\left(T_a\right)^k{}_\ell\delta^\ell{}_k \\ +\tfrac{4}{9}\alpha_s\alpha\left(T_b\right)^i{}_j\left(T_b\right)^\ell{}_k\delta^j{}_i\delta^k{}_\ell + \tfrac{16}{81}\alpha^2\delta^i{}_j\delta^j{}_i\delta^\ell{}_k\delta^k{}_\ell\end{matrix}\right] \\
&= \frac{4\pi}{27s}\left[\alpha_s^2\operatorname{Tr}\left(T_aT_b\right)\operatorname{Tr}\left(T_aT_b\right) + \tfrac{4}{9}\alpha_s\alpha\operatorname{Tr}\left(T_a\right)\operatorname{Tr}\left(T_a\right) + \tfrac{4}{9}\alpha_s\alpha\operatorname{Tr}\left(T_b\right)\operatorname{Tr}\left(T_b\right) + \tfrac{16}{81}\alpha^2\delta^i{}_i\delta^k{}_k\right] \\
&= \frac{4\pi}{27s}\left[\tfrac{1}{2}\cdot\tfrac{1}{2}\alpha_s^2\delta_{ab}\delta_{ab} + 0 + 0 + \tfrac{16}{9}\alpha^2\right] = \frac{4\pi}{27s}\left(2\alpha_s^2 + \tfrac{16}{9}\alpha^2\right).
\end{aligned}
$$

The fact that the cross-terms went away was apparent in the computation, and was a consequence of the simple identity $\operatorname{Tr}\left(T_a\right) = 0$.