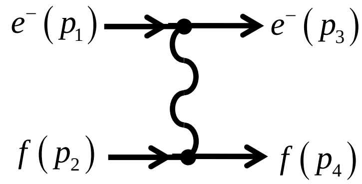
Solutions to Problems 7b

7. Find the differential cross-section for the process $e^- f \to e^- f$, treating both the electron and *f* as massless. Note that it diverges at $\theta = 0$. Explain this qualitatively. Calculate the total cross-section, assuming there is a minimum angle $\theta_{\min}$ at which the scattering can be observed.



There is only one Feynman diagram, sketched at right, which yields a Feynman amplitude of

$$i\mathcal{M} = \frac{-ig_{\mu\nu}(ie)(-ieQ)}{(p_1 - p_3)^2}\left(\bar{u}_3\gamma^\mu u_1\right)\left(\bar{u}_4\gamma^\mu u_2\right) = \frac{ie^2 Q}{2p_1 \cdot p_3}\left(\bar{u}_3\gamma^\mu u_1\right)\left(\bar{u}_4\gamma_\mu u_2\right),$$

$$(i\mathcal{M})^* = \frac{-ie^2 Q}{2p_1 \cdot p_3}\left(\bar{u}_1\gamma^\nu u_3\right)\left(\bar{u}_2\gamma_\nu u_4\right),$$

$$\left|i\mathcal{M}\right|^2 = \frac{e^4 Q^2}{4(p_1 \cdot p_3)^2}\left(\bar{u}_3\gamma^\mu u_1\right)\left(\bar{u}_1\gamma^\nu u_3\right)\left(\bar{u}_4\gamma_\mu u_2\right)\left(\bar{u}_2\gamma_\nu u_4\right).$$

Because we have two fermions in the initial state, we need to sum over all spin states and divide by four. We therefore have

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}\left|i\mathcal{M}\right|^2 &= \frac{e^4 Q^2}{16(p_1 \cdot p_3)^2}\mathrm{Tr}\left(\not{p}_3\gamma^\mu \not{p}_1\gamma^\nu\right)\mathrm{Tr}\left(\not{p}_4\gamma_\mu \not{p}_2\gamma_\nu\right) \\
&= \frac{e^4 Q^2}{(p_1 \cdot p_3)^2}\left(p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - p_3 \cdot p_1 g^{\mu\nu}\right)\left(p_{4\mu}p_{2\nu} + p_{4\nu}p_{2\mu} - p_4 \cdot p_2 g_{\mu\nu}\right) \\
&= \frac{e^4 Q^2}{(p_1 \cdot p_3)^2}\begin{bmatrix}(p_3 \cdot p_4)(p_1 \cdot p_2)(1+1) + (p_3 \cdot p_2)(p_1 \cdot p_4)(1+1) \\ +(p_3 \cdot p_1)(p_4 \cdot p_2)(-1-1-1-1+4)\end{bmatrix} \\
&= \frac{2e^4 Q^2}{(p_1 \cdot p_3)^2}\left[(p_3 \cdot p_4)(p_1 \cdot p_2) + (p_3 \cdot p_2)(p_1 \cdot p_4)\right].
\end{aligned}$$

We now need to write in an explicit form for all the momenta. We have

$$p_1^\mu = (E, 0, 0, E), \quad p_3^\mu = (E, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta),$$
$$p_2^\mu = (E, 0, 0, -E), \quad p_4^\mu = (E, -E\sin\theta\cos\phi, -E\sin\theta\sin\phi, -E\cos\theta).$$

The dot products will then be

$$p_1 \cdot p_2 = p_3 \cdot p_4 = 2E^2, \quad p_1 \cdot p_4 = p_2 \cdot p_3 = E^2(1+\cos\theta), \quad p_1 \cdot p_3 = p_2 \cdot p_4 = E^2(1-\cos\theta).$$

Substituting this into the previous equation, we have

$$\frac{1}{4}\sum_{\text{spins}}\left|i\mathcal{M}\right|^2=\frac{2e^4Q^2}{E^4\left(1-\cos\theta\right)^2}\left[4E^4+E^4\left(1+\cos\theta\right)^2\right]=\frac{2e^4Q^2}{\left(1-\cos\theta\right)^2}\left(\cos^2\theta+2\cos\theta+5\right).$$

We now work in a straightforward and familiar manner towards the cross section. We find

$$\sigma=\frac{D}{4\left|E_2\mathbf{p}_1-E_2\mathbf{p}_1\right|}=\frac{1}{8E^2}\cdot\frac{\left|\mathbf{p}_3\right|}{16\pi^2\left(2E\right)}\int\frac{1}{4}\sum_{\text{spins}}\left|i\mathcal{M}\right|^2d\Omega=\frac{2e^4Q^2}{256\pi^2E^2}\int\frac{\cos^2\theta+2\cos\theta+5}{\left(1-\cos\theta\right)^2}d\Omega\,,$$

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2Q^2}{8E^2}\cdot\frac{\cos^2\theta+2\cos\theta+5}{\left(1-\cos\theta\right)^2}.$$

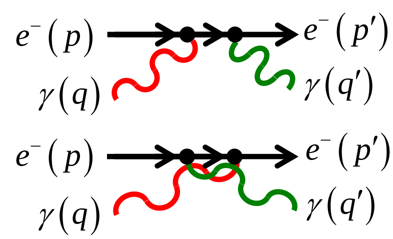
The cross-section is then infinite in the forward direction. This represents the fact that even when the electron and fermion miss each other by a large distance, they will still scatter, albeit at a small angle. We tame this infinity by assuming there is some minimum angle θ_{min} where the scattering can be detected. We then proceed to perform the remaining integrals to find the total cross section. In the computation below, we make the substitution $\cos\theta=1-z$ to make the integral easier.

$$\begin{aligned}\sigma&=\frac{\alpha^2Q^2}{8E^2}\int_0^{2\pi}d\phi\int_{-1}^{\cos\theta_{\text{min}}}\frac{\cos^2\theta+2\cos\theta+5}{\left(1-\cos\theta\right)^2}d\cos\theta=\frac{\pi\alpha^2Q^2}{4E^2}\int_2^{1-\cos\theta_{\text{min}}}\frac{\left(1-z\right)^2+2\left(1-z\right)+5}{\left(-z\right)^2}d\left(-z\right)\\&=\frac{\pi\alpha^2Q^2}{4E^2}\int_{1-\cos\theta_{\text{min}}}^{2}\left(1-4z^{-1}+8z^{-2}\right)dz=\frac{\pi\alpha^2Q^2}{4E^2}\left(z-4\ln z-8z^{-1}\right)\Big|_{1-\cos\theta_{\text{min}}}^{2}\\&=\frac{\pi\alpha^2Q^2}{4E^2}\left[2-1+\cos\theta_{\text{min}}+4\ln\left(\frac{1-\cos\theta_{\text{min}}}{2}\right)-4+\frac{8}{1-\cos\theta_{\text{min}}}\right].\end{aligned}$$

This rather horrendous result can be made a bit more palatable using the equation $\cos\theta=1-2\sin^2\left(\tfrac{1}{2}\theta\right)$, to yield

$$\sigma=\frac{\pi\alpha^2Q^2}{E^2}\left[\cot^2\left(\tfrac{1}{2}\theta_{\text{min}}\right)+\tfrac{1}{2}\cos^2\left(\tfrac{1}{2}\theta_{\text{min}}\right)+\ln\left(\sin\left(\tfrac{1}{2}\theta_{\text{min}}\right)\right)\right].$$

9. Write the complete Feynman amplitude for the process $e^-(p)\gamma(q)\to e^-(p')\gamma(q')$. Then make the approximation that the electron is massless. Calculate the differential cross-section. The total cross-section will come out infinite, but this is an artifact of treating the electron as massless.



There are two relevant Feynman diagrams, sketched at right. The resulting Feynman amplitude is then given by

$$i\mathcal{M}=(ie)^2\, i\varepsilon_\mu\varepsilon_\nu'^{*}\left[\bar{u}'\gamma^\nu\frac{\not{p}+\not{q}+m}{(p+q)^2-m^2}\gamma^\mu u+\bar{u}'\gamma^\mu\frac{\not{p}'-\not{q}+m}{(p'-q)^2-m^2}\gamma^\nu u\right]$$

$$=-\frac{ie^2}{2}\varepsilon_\mu\varepsilon_\nu'^{*}\bar{u}'\left[\gamma^\nu\frac{\not{p}+\not{q}+m}{p\cdot q}\gamma^\mu+\gamma^\mu\frac{\not{q}-\not{p}'-m}{p'\cdot q}\gamma^\nu\right]u$$

We then take the massless limit by simply setting $m = 0$. We then square the amplitude to yield

$$|i\mathcal{M}|^2=\frac{e^4}{4}\varepsilon_\mu\varepsilon_\alpha^{*}\varepsilon_\nu'^{*}\varepsilon_\beta'\bar{u}'\left[\gamma^\nu\frac{\not{p}+\not{q}}{p\cdot q}\gamma^\mu+\gamma^\mu\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma^\nu\right]u\,\bar{u}\left[\gamma^\alpha\frac{\not{p}+\not{q}}{p\cdot q}\gamma^\beta+\gamma^\beta\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma^\alpha\right]u'.$$

We now want to average this over the initial spins and polarizations, and sum over the final spins and polarizations, to yield

$$\frac{1}{4}\sum|i\mathcal{M}|^2=\frac{e^4}{16}g_{\mu\alpha}g_{\nu\beta}\mathrm{Tr}\left\{\not{p}'\left[\gamma^\nu\frac{\not{p}+\not{q}}{p\cdot q}\gamma^\mu+\gamma^\mu\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma^\nu\right]\not{p}\left[\gamma^\alpha\frac{\not{p}+\not{q}}{p\cdot q}\gamma^\beta+\gamma^\beta\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma^\alpha\right]\right\}$$

$$=\frac{e^4}{16}\mathrm{Tr}\left\{\not{p}'\left[\gamma^\nu\frac{\not{p}+\not{q}}{p\cdot q}\gamma^\mu+\gamma^\mu\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma^\nu\right]\not{p}\left[\gamma_\mu\frac{\not{p}+\not{q}}{p\cdot q}\gamma_\nu+\gamma_\nu\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma_\mu\right]\right\}$$

$$=\frac{e^4}{16}\mathrm{Tr}\left\{\begin{aligned}&\not{p}'\gamma^\nu\frac{\not{p}+\not{q}}{p\cdot q}\gamma^\mu\not{p}\gamma_\mu\frac{\not{p}+\not{q}}{p\cdot q}\gamma_\nu+\not{p}'\gamma^\nu\frac{\not{p}+\not{q}}{p\cdot q}\gamma^\mu\not{p}\gamma_\nu\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma_\mu\\&+\not{p}'\gamma^\mu\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma^\nu\not{p}\gamma_\mu\frac{\not{p}+\not{q}}{p\cdot q}\gamma_\nu+\not{p}'\gamma^\mu\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma^\nu\not{p}\gamma_\nu\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma_\mu\end{aligned}\right\}$$

We now start using the identities with pairs of Dirac matrices to simplify this a lot. We also use simplifications $\not{p}^2=p^2=0=p'^2=\not{p}'^2$.

$$\frac{1}{4}\sum|i\mathcal{M}|^2=-\frac{e^4}{8}\mathrm{Tr}\left\{\begin{array}{l}\not{p}'\gamma^\nu\frac{\not{p}+\not{q}}{p\cdot q}\not{p}\frac{\not{p}+\not{q}}{p\cdot q}\gamma_\nu+\not{p}'\gamma^\nu\frac{\not{p}+\not{q}}{p\cdot q}\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma_\nu\not{p}\\+\not{p}'\not{p}\gamma^\nu\frac{\not{q}-\not{p}'}{p'\cdot q}\frac{\not{p}+\not{q}}{p\cdot q}\gamma_\nu+\not{p}'\gamma^\mu\frac{\not{q}-\not{p}'}{p'\cdot q}\not{p}\frac{\not{q}-\not{p}'}{p'\cdot q}\gamma_\mu\end{array}\right\}$$

$$=\frac{e^4}{4}\mathrm{Tr}\left\{\begin{array}{l}\not{p}'\frac{\not{p}+\not{q}}{p\cdot q}\not{p}\frac{\not{p}+\not{q}}{p\cdot q}-2\not{p}'\frac{(p+q)\cdot(q-p')}{(p\cdot q)(p'\cdot q)}\not{p}\\-2\not{p}'\not{p}\frac{(q-p')\cdot(p+q)}{(p\cdot q)(p'\cdot q)}+\not{p}'\frac{\not{q}-\not{p}'}{p'\cdot q}\not{p}\frac{\not{q}-\not{p}'}{p'\cdot q}\end{array}\right\}$$

$$=\frac{e^4}{4}\mathrm{Tr}\left\{\frac{1}{(p\cdot q)^2}\not{p}'\not{q}\not{p}\not{q}-4\frac{p\cdot q-p'\cdot q-p\cdot p'}{(p\cdot q)(p'\cdot q)}\not{p}'\not{p}+\frac{1}{(p'\cdot q)^2}\not{p}'\not{q}\not{p}\not{q}\right\}$$

$$=2e^4\left\{\frac{(p\cdot q)(p'\cdot q)}{(p\cdot q)^2}-\frac{(p+q-p')^2(p\cdot p')}{(p\cdot q)(p'\cdot q)}+\frac{(p\cdot q)(p'\cdot q)}{(p'\cdot q)^2}\right\}=2e^4\left(\frac{p'\cdot q}{p\cdot q}+\frac{p\cdot q}{p'\cdot q}\right).$$

We now write out the momenta in the usual way, so we have

$$p^\mu=(E,0,0,E),\quad q^\mu=(E,0,0,-E),\quad p'^\mu=(E,E\sin\theta\cos\phi,E\sin\theta\sin\phi,E\cos\theta).$$

The relevant dot products are then

$$p\cdot q=2E^2,\quad p'\cdot q=E^2(1+\cos\theta).$$

Substituting in, we then have

$$\frac{1}{4}\sum|i\mathcal{M}|^2=e^4\left(1+\cos\theta+\frac{4}{1+\cos\theta}\right).$$

It is interesting that in this case the apparent infinity comes from the *backwards* direction, where the electron reverses direction. As mentioned, this occurs because of the approximation. The troublemaking propagator denominator is $p'\cdot q$, which can easily be shown to not vanish if we include masses.

We work our way to the differential cross section to have

$$\sigma=\frac{D}{8E^2}=\frac{1}{8E^2}\cdot\frac{E}{16\pi^2 2E}\int\frac{1}{4}\sum|i\mathcal{M}|^2 d\Omega=\frac{e^4}{256\pi^2E^2}\int\left(1+\cos\theta+\frac{4}{1+\cos\theta}\right)d\Omega,$$

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{16E^2}\left(1+\cos\theta+\frac{4}{1+\cos\theta}\right).$$