Solutions to Problems 6b

6. Starting from eq. (6.26), work out the differential and total cross-section for $\psi(p_1)\bar{\psi}(p_2)\to\psi(p_3)\bar{\psi}(p_4)$ in the limit $m=M=0$. The result should be very simple, and should be easy to integrate over angles (if it isn't, you made an error somewhere). For an added challenge, let $m\neq 0$ but keep $M=0$.

We'll skip the added challenge. We have

$$i\mathcal{M}=\frac{ig^2(\bar{u}_3\gamma_5u_1)(\bar{v}_2\gamma_5v_4)}{(p_1-p_3)^2-M^2}-\frac{ig^2(\bar{v}_2\gamma_5u_1)(\bar{u}_3\gamma_5v_4)}{(p_1+p_2)^2-M^2}=ig^2\left[\frac{(\bar{u}_3\gamma_5u_1)(\bar{v}_2\gamma_5v_4)}{p_1^2+p_3^2-2p_1\cdot p_3}-\frac{(\bar{v}_2\gamma_5u_1)(\bar{u}_3\gamma_5v_4)}{p_1^2+p_2^2+2p_1\cdot p_2}\right]$$
$$=-\frac{ig^2}{2}\left[\frac{(\bar{u}_3\gamma_5u_1)(\bar{v}_2\gamma_5v_4)}{p_1\cdot p_3}+\frac{(\bar{v}_2\gamma_5u_1)(\bar{u}_3\gamma_5v_4)}{p_1\cdot p_2}\right]$$

The Hermitian conjugate of this is

$$(i\mathcal{M})^*=\frac{ig^2}{2}\left[\frac{(\bar{u}_1\gamma_5u_3)(\bar{v}_4\gamma_5v_2)}{p_1\cdot p_3}+\frac{(\bar{u}_1\gamma_5v_2)(\bar{v}_4\gamma_5u_3)}{p_1\cdot p_2}\right].$$

Let's multiply these together, then let's try to multiply it out, always putting things together that will be combined when we turn it into sums and traces. We have

$$|i\mathcal{M}|^2=\frac{g^4}{4}\left[\frac{(\bar{u}_3\gamma_5u_1)(\bar{v}_2\gamma_5v_4)}{p_1\cdot p_3}+\frac{(\bar{v}_2\gamma_5u_1)(\bar{u}_3\gamma_5v_4)}{p_1\cdot p_2}\right]\left[\frac{(\bar{u}_1\gamma_5u_3)(\bar{v}_4\gamma_5v_2)}{p_1\cdot p_3}+\frac{(\bar{u}_1\gamma_5v_2)(\bar{v}_4\gamma_5u_3)}{p_1\cdot p_2}\right]$$
$$=\frac{g^4}{4}\left[\begin{array}{l}\frac{(\bar{u}_3\gamma_5u_1\bar{u}_1\gamma_5u_3)(\bar{v}_2\gamma_5v_4\bar{v}_4\gamma_5v_2)}{(p_1\cdot p_3)^2}+\frac{(\bar{u}_3\gamma_5u_1\bar{u}_1\gamma_5v_2\bar{v}_2\gamma_5v_4\bar{v}_4\gamma_5u_3)}{(p_1\cdot p_3)(p_1\cdot p_2)}\\+\frac{(\bar{v}_2\gamma_5u_1\bar{u}_1\gamma_5u_3\bar{u}_3\gamma_5v_4\bar{v}_4\gamma_5v_2)}{(p_1\cdot p_3)(p_1\cdot p_2)}+\frac{(\bar{v}_2\gamma_5u_1\bar{u}_1\gamma_5v_2)(\bar{u}_3\gamma_5v_4\bar{v}_4\gamma_5u_3)}{(p_1\cdot p_2)^2}\end{array}\right].$$

We now sum on final spins and average over initial spins and rewrite everything as traces, so we have

$$\frac{1}{4}\sum_{\text{spins}}|i\mathcal{M}|^2=\frac{g^4}{16}\left[\begin{array}{l}\frac{\mathrm{Tr}(\not{p}_3\gamma_5\not{p}_1\gamma_5)\mathrm{Tr}(\not{p}_2\gamma_5\not{p}_4\gamma_5)}{(p_1\cdot p_3)^2}+\frac{\mathrm{Tr}(\not{p}_3\gamma_5\not{p}_1\gamma_5\not{p}_2\gamma_5\not{p}_4\gamma_5)}{(p_1\cdot p_3)(p_1\cdot p_2)}\\+\frac{\mathrm{Tr}(\not{p}_2\gamma_5\not{p}_1\gamma_5\not{p}_3\gamma_5\not{p}_4\gamma_5)}{(p_1\cdot p_3)(p_1\cdot p_2)}+\frac{\mathrm{Tr}(\not{p}_2\gamma_5\not{p}_1\gamma_5)\mathrm{Tr}(\not{p}_3\gamma_5\not{p}_4\gamma_5)}{(p_1\cdot p_2)^2}\end{array}\right].$$

We then anti-commute the γ_5's together and let them annihilate each other, which then gives us

$$\frac{1}{4}\sum_{\text{spins}}|i\mathcal{M}|^2 = \frac{g^4}{16}\left[\frac{\text{Tr}(\not{p}_3\not{p}_1)\text{Tr}(\not{p}_2\not{p}_4)}{(p_1\cdot p_3)^2}+\frac{\text{Tr}(\not{p}_3\not{p}_1\not{p}_2\not{p}_4)}{(p_1\cdot p_3)(p_1\cdot p_2)} + \frac{\text{Tr}(\not{p}_2\not{p}_1\not{p}_3\not{p}_4)}{(p_1\cdot p_3)(p_1\cdot p_2)}+\frac{\text{Tr}(\not{p}_2\not{p}_1)\text{Tr}(\not{p}_3\not{p}_4)}{(p_1\cdot p_2)^2}\right]$$

$$= g^4\left[\frac{(p_3\cdot p_1)(p_2\cdot p_4)}{(p_1\cdot p_3)^2}+\frac{(p_3\cdot p_1)(p_2\cdot p_4)+(p_3\cdot p_4)(p_1\cdot p_2)-(p_3\cdot p_2)(p_1\cdot p_4)}{4(p_1\cdot p_3)(p_1\cdot p_2)} + \frac{(p_1\cdot p_2)(p_3\cdot p_4)+(p_3\cdot p_1)(p_2\cdot p_4)-(p_3\cdot p_2)(p_1\cdot p_4)}{4(p_1\cdot p_3)(p_1\cdot p_2)}+\frac{(p_2\cdot p_1)(p_3\cdot p_4)}{(p_1\cdot p_2)^2}\right]$$

We note that the two middle terms are identical. If we now let the incoming and outgoing four-momenta be

$$p_1^\mu = (E,0,0,E),\quad p_3^\mu = (E, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta),$$
$$p_2^\mu = (E,0,0,-E),\quad p_4^\mu = (E, -E\sin\theta\cos\phi, -E\sin\theta\sin\phi, -E\cos\theta),$$

then it isn't hard to work out all the relevant dot products:

$$p_1\cdot p_2 = p_3\cdot p_4 = 2E^2,\quad p_1\cdot p_3 = p_2\cdot p_4 = E^2(1+\cos\theta),\quad p_1\cdot p_4 = p_2\cdot p_3 = E^2(1-\cos\theta).$$

We now substitute this into our mess, and we have

$$\frac{1}{4}\sum_{\text{spins}}|i\mathcal{M}|^2 = g^4\left[\frac{E^4(1-\cos\theta)^2}{E^4(1-\cos\theta)^2}+\frac{E^4(1-\cos\theta)^2+4E^4-E^4(1+\cos\theta)^2}{2\cdot 2E^4(1-\cos\theta)}+\frac{4E^4}{4E^4}\right]$$
$$= g^4\left[1+\frac{1-2\cos\theta+\cos^2\theta+4-1-2\cos\theta-\cos^2\theta}{4-4\cos\theta}+1\right] = g^4(1+1+1) = 3g^4.$$

And suddenly, for no reason, the formulas suddenly become very simple. We now proceed quickly to the cross-section, which is given by

$$\sigma = \frac{D}{4\left(E^2+E^2\right)} = \frac{1}{8E^2}\frac{p}{16\pi^2 E_{cm}}\int\frac{1}{4}\sum_{\text{spins}}|i\mathcal{M}|^2 d\Omega = \frac{E}{128\pi^2E^2(2E)}\int 3g^4 d\Omega = \frac{3g^4}{256\pi^2E^2}\int d\Omega.$$

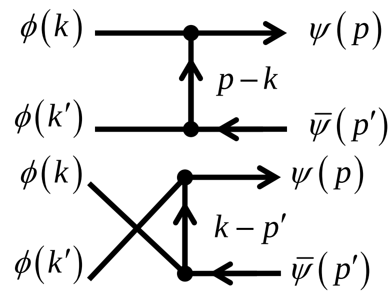
The differential and total cross sections are therefore

$$\frac{d\sigma}{d\Omega} = \frac{3g^4}{256\pi^2E^2},\quad \sigma = \frac{3g^4}{64\pi E^2}.$$

Of course, this can be trivially rewritten in terms of $s = 4E^2$, if we want, so that

$$\frac{d\sigma}{d\Omega} = \frac{3g^4}{64\pi^2 s},\quad \sigma = \frac{3g^4}{16\pi s}.$$

7. Write the full Feynman amplitude for $\phi(k)\phi(k') \to \psi(p)\bar\psi(p')$ for pseudoscalar couplings. Show it can be simplified to an expression of the form $(\bar u \not k v') f$, where *f* is some function of the momenta. Work out the differential cross-section $d\sigma/d\Omega$ in the cm frame.



The Feynman diagrams are listed above right, with the correct intermediate momentum marked in each case. I have in each case written it in terms of *k*, in anticipating of writing our amplitudes in terms of *k*. The two diagrams differ by the swapping of an external boson line, and hence get a relative plus sign. The corresponding Feynman amplitude is then

$$\begin{aligned} i\mathcal{M} &= \bar u g\gamma_5 \frac{i(\not p - \not k + m)}{(p-k)^2 - m^2} g\gamma_5 v' + \bar u g\gamma_5 \frac{i(\not k - \not p' + m)}{(k-p')^2 - m^2} g\gamma_5 v' \\ &= ig^2\left[\frac{\bar u\gamma_5(\not p - \not k + m)\gamma_5 v'}{p^2 + k^2 - 2p\cdot k - m^2} + \frac{\bar u\gamma_5(\not k - \not p' + m)\gamma_5 v'}{p'^2 + k^2 - 2p'\cdot k - m^2}\right] \\ &= ig^2\left[\frac{\bar u(\not k - \not p + m)\gamma_5\gamma_5 v'}{m^2 + M^2 - 2p\cdot k - m^2} + \frac{\bar u\gamma_5\gamma_5(\not p' - \not k + m)v'}{m^2 + M^2 - 2p'\cdot k - m^2}\right] \\ &= ig^2\left[\frac{\bar u\not k v'}{M^2 - 2p\cdot k} - \frac{\bar u \not k v'}{M^2 - 2p'\cdot k}\right] = ig^2 \bar u \not k v'\left(\frac{1}{M^2 - 2p\cdot k} - \frac{1}{M^2 - 2p'\cdot k}\right). \end{aligned}$$

We now want to sum this over final spins after we square it, so we have

$$\begin{aligned} \sum_{s,s'} |i\mathcal{M}|^2 &= (ig^2)(-ig^2)\left(\frac{1}{M^2 - 2p\cdot k} - \frac{1}{M^2 - 2p'\cdot k}\right)^2 \sum_{s,s'}(\bar u \not k v' \bar v' \not k u) \\ &= g^4\left(\frac{1}{M^2 - 2p\cdot k} - \frac{1}{M^2 - 2p'\cdot k}\right)^2 \mathrm{Tr}\left[(\not p + m)\not k(\not p' - m)\not k\right] \\ &= g^4\left(\frac{1}{M^2 - 2p\cdot k} - \frac{1}{M^2 - 2p'\cdot k}\right)^2 \mathrm{Tr}\left[\not p \not k \not p' \not k - m^2 \not k^2\right] \\ &= 4g^4\left(\frac{1}{M^2 - 2p\cdot k} - \frac{1}{M^2 - 2p'\cdot k}\right)^2 \left[(p\cdot k)(p'\cdot k) + (p'\cdot k)(p\cdot k) - (p\cdot p')k^2 - m^2k^2\right]. \end{aligned}$$

We now need to work out all the dot products. Working in the center of mass frame, since the initial particles have matching masses they will have massing energies *E*, and so will the final particles. We let *k* and *p* stand for the initial and final momenta respectively. Then the four momenta will be given by

$$\begin{aligned} k^\mu &= (E, 0, 0, k), \quad p^\mu = (E, p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta), \\ k'^\mu &= (E, 0, 0, -k), \quad p'^\mu = (E, -p\sin\theta\cos\phi, -p\sin\theta\sin\phi, -p\cos\theta). \end{aligned}$$

The dot products we need are therefore

$$p \cdot k = E^2 - pk\cos\theta\,, \quad p' \cdot k = E^2 + pk\cos\theta\,, \quad p \cdot p' = E^2 + p^2\,.$$

Substituting this in, we have

$$\begin{aligned}\sum_{s,s'}\left|i\mathcal{M}\right|^2 &= 4g^4\left(\frac{1}{M^2 - 2E^2 + 2pk\cos\theta} - \frac{1}{M^2 - 2E^2 - 2pk\cos\theta}\right)^2 \\ &\quad\times\left[2\left(E^2 - pk\cos\theta\right)\left(E^2 + pk\cos\theta\right) - \left(E^2 + p^2 + m^2\right)M^2\right] \\ &= \frac{128g^4 p^2k^2\cos^2\theta\left(E^4 - p^2k^2\cos^2\theta - E^2M^2\right)}{\left[\left(M^2 - 2E^2\right)^2 - 4p^2k^2\cos^2\theta\right]^2} = \frac{128g^4 p^2k^4\cos^2\theta\left(E^2 - p^2\cos^2\theta\right)}{\left[\left(2E^2 - M^2\right)^2 - 4p^2k^2\cos^2\theta\right]^2}\,.\end{aligned}$$

We then substitute this into our standard formulas to get the cross-section, but do not perform the final integral.

$$\sigma = \frac{D}{4(2Ek)} = \frac{1}{8Ek}\frac{p}{16\pi^2(2E)}\int\sum_{s,s'}\left|i\mathcal{M}\right|^2 = \frac{p}{256\pi^2E^2k}\int\frac{128g^4p^2k^4\cos^2\theta\left(E^2 - p^2\cos^2\theta\right)}{\left[\left(2E^2 - M^2\right)^2 - 4p^2k^2\cos^2\theta\right]^2}d\Omega\,,$$

$$\frac{d\sigma}{d\Omega} = \frac{g^4p^3k^3\cos^2\theta\left(E^2 - p^2\cos^2\theta\right)}{2\pi^2E^2\left[\left(2E^2 - M^2\right)^2 - 4p^2k^2\cos^2\theta\right]^2}\,.$$

We were not asked to find the cross-section, and I don’t like the look of that integral. If we wish to write everything in terms of the energy, this is trivial, since

$$p^2 = E^2 - m^2\,, \quad k^2 = E^2 - M^2\,.$$