Solutions to Problems 5b

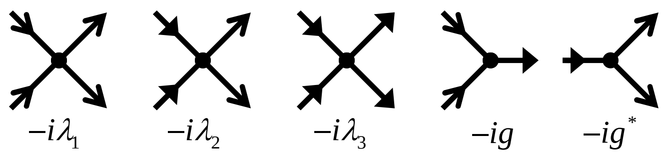
11. In problem 4.6c, you found all relevant matrix elements for a theory with two particles, one of which had twice the charge of the other. Make up a notation for the two particles and their corresponding anti-particles and give me a complete list of Feynman rules: propagators and couplings. Let m_1 be the mass of ψ_1 and m_2 be the mass of ψ_2. Name the Hamiltonian matrix elements as:

$$\langle 0|\mathcal{H}|\psi_2^*\psi_1\psi_1\rangle = g\,, \quad \langle 0|\mathcal{H}|\psi_2\psi_1^*\psi_1^*\rangle = g^*\,,$$

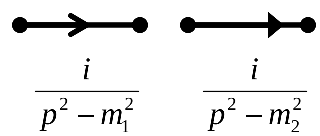
$$\langle 0|\mathcal{H}|\psi_1\psi_1\psi_1^*\psi_1^*\rangle = \lambda_1\,, \quad \langle 0|\mathcal{H}|\psi_1\psi_2\psi_1^*\psi_2^*\rangle = \lambda_2\,, \quad \langle 0|\mathcal{H}|\psi_2\psi_2\psi_2^*\psi_2^*\rangle = \lambda_3\,.$$

We denote ψ_1 by an open arrow and ψ_2 by a closed arrow. Then the rules are straightforward.

- For each incoming or outgoing ψ_1 (ψ_2) draw an open (closed) arrow on the left, or an open (closed) arrow on the right.
  For each $\bar{\psi}_1$ ($\bar{\psi}_2$) draw the arrows the other direction.
- Draw all topologically distinct diagram connecting the initial states to the final state, using the five vertices sketched at right.



- Conserve four-momentum at all vertices
- For each vertex, include the factor listed in the diagram above
- For each interior line with an open arrow, include a factor of $i/(p^2-m_1^2)$.
- For each interior line with a closed arrow, include a factor of $i/(p^2-m_2^2)$.



- Multiply all factors for each diagram. Then add the contribution from all diagrams.

12. Using the Feynman rules from problem 11, calculate the decay rate for $\psi_2 \to \psi_1\psi_1$ in this theory. What inequality must be true for this decay to occur? Can the ψ_1 particle be unstable in this theory?

There is only one diagram, denoted $-ig^*$ in the previous list of Feynman rules, and the Feynman amplitude is $i\mathcal{M} = -ig^*$. We have identical particles in the final state, so this will introduce a factor of ½ into the answer. We therefore have

$$\Gamma = \frac{D}{2m_2} = \frac{1}{2m_2}\frac{p}{16\pi^2 E_{cm}}\frac{1}{2}\int |i\mathcal{M}|^2\, d\Omega = \frac{4\pi p\left|-ig^*\right|^2}{64\pi^2 m_2^2} = \frac{4\pi p|g|^2}{64\pi^2 m_2^2}\,.$$

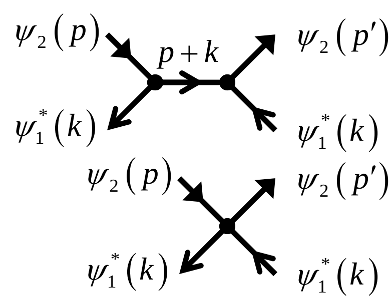
The initial mass m_2 is split between the two final state particles, and therefore each one has energy $\frac{1}{2}m_2$ and momentum $p=\sqrt{\frac{1}{4}m_2^2-m_1^2}$. Substituting this in, we have

$$\Gamma=\frac{|g|^2\sqrt{m_2^2-4m_1^2}}{32\pi m_2^2} .$$

This decay is only possible if $m_2>2m_1$.

Because of charge conservation, any ψ_1 decay must have at least one ψ_1 or ψ_1^* particle in the final state. Therefore, the final state will always be heavier than the initial state, which is impossible.

13. Using the Feynman rules from problem 11, calculate the cross-section $\psi_2\psi_1^*\to\psi_2\psi_1^*$. Is there any chance that there will be resonance?



We arbitrarily name the initial state momenta *p* and *k*, and then the final state momenta p' and k' . The relevant Feynman diagrams are sketched at right. The internal momentum for the first diagram is $p+k$. The Feynman amplitude for these two processes together is then just

$$i\mathcal{M}=\frac{(-ig)(-ig^*)i}{(p+k)^2-m_1^2}-i\lambda_2=-i\frac{|g|^2}{s-m_1^2}-i\lambda_2 .$$

We have taken advantage of the fact that since the initial momenta are *p* and *k*, we have $s=(p+k)^2$, which is independent of angle. We then find the cross-section using

$$\sigma=\frac{D}{4|E_k\mathbf{p}-E_p\mathbf{k}|}=\frac{1}{4(E_k+E_p)p}\frac{p'}{16\pi^2E_{cm}}\int d\Omega|i\mathcal{M}|^2=\frac{1}{64\pi^2E_{cm}^2}\int d\Omega|i\mathcal{M}|^2 .$$

It is a little tricky to note that since the final two masses match the initial two masses, we must have the magnitude of the final momenta match that of the initial momenta in the cm frame, since this makes the energy work out. We note also that $s=E_{cm}^2$, and the integral is independent of angle, so we conclude

$$\sigma=\frac{1}{16\pi s}\left(\frac{|g|^2}{s-m_1^2}+\lambda_2\right)^2 .$$

This answer could be rewritten in terms of other variables, such as the magnitude of the momenta, but this form is the simplest. We note that resonance would occur whenever $m_1=\sqrt{s}=E_{cm}=E_p+E_k$, but $E_p=\sqrt{m_1^2+\mathbf{p}^2}>m_1$, so resonance is never a problem.