Solutions to Problems 2b

7. [10] Suppose an electron/positron collider collides beams with energies E_1 and E_2 head on. What is *s*? Treat the electron and positrons as massless. If the BABAR experiment is trying to create the $\Upsilon(4s)$ resonance with mass $M = 10.58\text{ GeV}$ by colliding electrons with energy $E_1 = 9.00\text{ GeV}$ electrons, what energy must the positrons be?

To make things simple, let's collide them coming in on the x^3-axis. Then since they are being treated as massless, the momenta are equal to the energies. Hence the four-momenta are

$$p_1 = (E_1, 0, 0, E_1), \quad p_2 = (E_2, 0, 0, -E_2).$$

The total initial momentum and *s* are given by

$$p_1 + p_2 = (E_1 + E_2, 0, 0, E_1 - E_2),$$

$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (E_1 - E_2)^2 = E_1^2 + 2E_1E_2 + E_2^2 - E_1^2 + 2E_1E_2 - E_2^2 = 4E_1E_2 .$$

To create the $\Upsilon(4s)$ resonance, we must have sufficient energy, so that $s = M^2$. We therefore have

$$E_2 = \frac{s}{4E_1} = \frac{M^2}{4E_1} = \frac{(10.58\text{ GeV})^2}{4(9.00\text{ GeV})} = 3.11\text{ GeV} .$$

8. [10] A particle of mass *M* decays to two particles. Find a general formula for the magnitude of the final three-momentum:

(a) [5] If the mass of each final particle is *m*;

(b) [5] If the mass of one final particle is *m* and the other is 0; and

(c) If the mass of the final particles are m_1 and m_2, and check that it leads to the correct results for parts (a) and (b).

Let the momenta of the initial particle be *p* and let the momenta of the final particles be p_1 and p_2. Then conservation of momentum tells us that $p = p_1 + p_2$. We rearrange this to $p - p_1 = p_2$, and then square it.

$$p_2^2 = (p - p_1)^2 , \quad m_2^2 = p^2 - 2p \cdot p_1 + p_1^2 , \quad m_2^2 = M^2 - 2p \cdot p_1 + m_1^2 .$$

We now write out the explicit form of the initial momentum and the momentum p_1, which gives us

$$p = (M, 0, 0, 0) \quad \text{and} \quad p_1 = (E_1, p_1 \sin\theta\cos\phi, p_1 \sin\theta\sin\phi, p_1\cos\theta), \quad p \cdot p_1 = ME_1 - 0 = ME_1 .$$

Substituting this in then give us

$$m_2^2 = M^2 - 2ME_1 + m_1^2 \quad ,$$
$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}$$

We can now quickly work out the answers to each of the parts. For part (a), $m_1 = m_2 = m$, and this simplifies to $E_1 = \tfrac{1}{2} M$, and we then find the momentum from

$$p_1 = \sqrt{E_1^2 - m_1^2} = \sqrt{\tfrac{1}{4} M^2 - m^2} \ .$$

For part (b), we can pick the first particle to be massless, so we have

$$p_1 = E_1 = \frac{M^2 - m^2}{2M} \, .$$

For part (c) we simply start with the most general expression and deal with the resulting mess.

$$p_1^2 = E_1^2 - m_1^2 = \left(\frac{M^2 + m_1^2 - m_2^2}{2M} \right)^2 - m_1^2 = \frac{1}{4M^2} \begin{pmatrix} M^4 + m_1^4 + m_2^4 + 2M^2 m_1^2 \\ -2M^2 m_2^2 - 2m_1^2 m_2^2 \end{pmatrix} - \frac{1}{4M^2} \left(4m_1^2 M^2 \right) ,$$
$$p_1 = \frac{1}{2M} \sqrt{M^4 + m_1^4 + m_2^4 - 2M^2 m_1^2 - 2M^2 m_2^2 - 2m_1^2 m_2^2} \ .$$

Since it wasn’t assigned, I’m not going to show how it simplifies to the expressions we found before.

9. [10] For any process where two particles of momenta p_1 and p_2 collide to make two final particles with momenta p_3 and p_4, define the *Mandelstam variables* by

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 , \quad t = (p_1 - p_3)^2 = (p_2 - p_4)^2 , \quad u = (p_1 - p_4)^2 = (p_2 - p_3)^2 .$$

Show that $s + t + u$ is a constant, and determine it in terms of the masses $m_i^2 = p_i^2$.

The expressions given are all equal to each other since we have, from conservation of four-momentum, $p_1 + p_2 = p_3 + p_4$. If you rearrange this in various ways and square it, you can prove all the pairs match. We square these expressions out to yield

$$s = m_1^2 + m_2^2 + 2 p_1 \cdot p_2 = m_3^2 + m_4^2 + 2 p_3 \cdot p_4 ,$$
$$t = m_1^2 + m_3^2 - 2 p_1 \cdot p_3 = m_2^2 + m_4^2 - 2 p_2 \cdot p_4 ,$$
$$u = m_1^2 + m_4^2 - 2 p_1 \cdot p_4 = m_2^2 + m_3^2 - 2 p_2 \cdot p_3 .$$

Now, it is not obvious how to proceed, but the quickest way is to take the conservation of momentum rule and solve it for any one of the momenta, say $p_4 = p_1 + p_2 - p_3$. Squaring,

$$p_4^2 = (p_1 + p_2 - p_3)^2 ,$$
$$m_4^2 = m_1^2 + m_2^2 + m_3^2 + 2 p_1 \cdot p_2 - 2 p_1 \cdot p_3 - 2 p_2 \cdot p_3$$
$$= m_1^2 + m_2^2 + m_3^2 + \left(s - m_1^2 - m_2^2\right) - \left(m_1^2 + m_3^2 - t\right) - \left(m_2^2 + m_3^2 - u\right)$$
$$= s + t + u - m_1^2 - m_2^2 - m_3^2 ,$$
$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 .$$

10. [15] The neutral Kaon system has two particles $\left|K_0\right\rangle$ and $\left|\bar{K}_0\right\rangle$. These particles are not mass eigenstates; they are related to mass eigenstates by

$$\left|K_0\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|K_1\right\rangle + \left|K_2\right\rangle\right), \quad \left|\bar{K}_0\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|K_1\right\rangle - \left|K_2\right\rangle\right).$$

These *are* eigenstate of the Hamiltonian, with energies $H\left|K_1\right\rangle = M_1\left|K_1\right\rangle$ and $H\left|K_2\right\rangle = M_2\left|K_2\right\rangle$. Suppose at $t=0$, we have $\left|\Psi(t=0)\right\rangle = \left|K_0\right\rangle$. What is $\left|\Psi(t)\right\rangle$ at all times? At time *t*, the particle is measured to see if it is a $\left|K_0\right\rangle$ or $\left|\bar{K}_0\right\rangle$. What is the probability of each of these? If $M_1 - M_2 = 3.484\times10^{-6}\text{ eV}$, at what time *t* will the particle first be 100% $\left|\bar{K}_0\right\rangle$?

The first step we need to do is to write the initial state in terms of the Hamiltonian eigenstates. This is pretty easy. We have

$$\left|\Psi(t=0)\right\rangle = \left|K_0\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|K_1\right\rangle + \left|K_2\right\rangle\right) .$$

Since the particles are at rest, these have energies $E_i = M_i$. Then using equation (2.43), the wave function at arbitrary time is given by

$$\left|\Psi(t)\right\rangle = \tfrac{1}{\sqrt{2}}\left(\left|K_1\right\rangle e^{-iM_1t} + \left|K_2\right\rangle e^{-iM_2t}\right) .$$

So far so good. We now want to know what the probability is at a later time that this is in the state $\left|\bar{K}_0\right\rangle$. This is given by

$$\begin{aligned} P\left(\bar{K}_0\right) &= \left|\left\langle \bar{K}_0 \middle| \Psi(t)\right\rangle\right|^2 = \left|\tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\left(\left\langle K_1\right| - \left\langle K_2\right|\right)\left(\left|K_1\right\rangle e^{-iM_1t} + \left|K_2\right\rangle e^{-iM_2t}\right)\right|^2 \\ &= \tfrac{1}{4}\left|e^{-iM_1t} - e^{-iM_2t}\right|^2 = \tfrac{1}{4}\left(e^{-iM_1t} - e^{-iM_2t}\right)^*\left(e^{-iM_1t} - e^{-iM_2t}\right) \\ &= \tfrac{1}{4}\left(e^{iM_1t} - e^{iM_2t}\right)\left(e^{-iM_1t} - e^{-iM_2t}\right) = \tfrac{1}{4}\left(1 - e^{iM_2t - iM_1t} - e^{iM_1t - iM_2t} + 1\right) \\ &= \tfrac{1}{4}\left(2 - 2\cos\left[\left(M_2 - M_1\right)t\right]\right) = \tfrac{1}{2} - \tfrac{1}{2}\cos\left[\left(M_1 - M_2\right)t\right]. \end{aligned}$$

It is not hard to show that similarly, $P\left(K_0\right) = \tfrac{1}{2} + \tfrac{1}{2}\cos\left[\left(M_1 - M_2\right)t\right]$. To get $P\left(\bar{K}_0\right) = 1$, we need the cosine to be – 1, which first occurs at π. We therefore have

$$t = \frac{\pi}{M_1 - M_2} = \frac{\pi\left(6.592\times10^{-16}\text{ eV}\cdot\text{s}\right)}{3.484\times10^{-6}\text{ eV}} = 5.94\times10^{-10}\text{ s} = 0.594\text{ ns} .$$

You may well wonder how such short distances are measured, but the particles are often moving relativistically, so we actually measure the distance and deduce the time.