Solutions to Problems 2a

1. [10] Find the determinant of each of the Lorentz transformations eqs. (2.6) and (2.7). Assuming the result is the same for all rotations and boosts, show that no combination of rotations and boosts can produce a parity transformation $\mathcal{P}$ nor a time reversal transformation $\mathcal{T}$.

The determinants can be found by starting with any row and multiplying each item in that row by the determinant of the "minor" after we delete the corresponding row and column, and introducing a minus sign whenever the combined row plus column number is odd. We therefore have

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1\cdot\det\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1\cdot\det\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\det\begin{pmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} = 1\cdot\det\begin{pmatrix} \cosh\phi & 0 & -\sinh\phi \\ 0 & 1 & 0 \\ -\sinh\phi & 0 & \cosh\phi \end{pmatrix}$$

$$= 1\cdot\det\begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} = \cosh^2\phi - \left(-\sinh\phi\right)^2$$

$$= \cosh^2\phi - \sin^2\phi = 1.$$

The determinants of parity and time reversal are easily found

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = 1\cdot\det\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1\cdot\det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1,$$

$$\det\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -1\cdot\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1\cdot\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1.$$

The determinant of any combination of rotations and boosts will the product of the determinants for each one, and therefore will be 1. Since parity and time reversal both have determinant -1, they must not be a combination of such operations.

2. [15] Simplify each of the following:

(a) [3] $g_{\alpha\beta}g^{\alpha\beta}$

$$g_{\alpha\beta}g^{\alpha\beta} = \delta_\alpha^\alpha = 4\,.$$

The trick here is to remember that there is an implied sum, so this is the sum of four terms.

(b) [7] $\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - p_1\cdot p_2 g^{\mu\nu}\right)\left(p_{3\mu}p_{4\nu} + p_{3\nu}p_{4\nu} - p_3\cdot p_4 g_{\mu\nu}\right)$

Whenever two factors are multiplied with one index up and one index down, or with a metric tensor *g* connecting them, it turns into a dot product, so we have

$$\begin{aligned}
&\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - p_1\cdot p_2 g^{\mu\nu}\right)\left(p_{3\mu}p_{4\nu} + p_{3\nu}p_{4\mu} - p_3\cdot p_4 g_{\mu\nu}\right)\\
&=\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)+\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right)-\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)\\
&+\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right)+\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)-\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)\\
&-\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)-\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)+\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)g^{\mu\nu}g_{\mu\nu}\\
&=\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)(1+1)+\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right)(1+1)+\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)(-1-1-1-1+4)\\
&=2\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)+2\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right).
\end{aligned}$$

(c) [5] $\varepsilon^{\alpha\mu\beta\nu}\varepsilon^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}p_{1\alpha}p_{2\beta}p_{3\rho}p_{4\tau}$

We'd like to use identity (2.17a), but this only works if the last two indices on the first Levi-Civita tensor match the last two on the second one. Fortunately, because the Levi-Civita tensor is completely anti-symmetric, we can rearrange the indices in any order we want if we just introduce a minus sign every time we switch a pair of indices, so that $\varepsilon^{\alpha\mu\beta\nu} = -\varepsilon^{\alpha\beta\mu\nu}$ and $\varepsilon^{\rho}{}_{\mu}{}^{\tau}{}_{\nu} = -\varepsilon^{\rho\tau}{}_{\mu\nu}$. We therefore have

$$\begin{aligned}
\varepsilon^{\alpha\mu\beta\nu}\varepsilon^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}p_{1\alpha}p_{2\beta}p_{3\rho}p_{4\tau} &= \varepsilon^{\alpha\beta\mu\nu}\varepsilon^{\rho\tau}{}_{\mu\nu}p_{1\alpha}p_{2\beta}p_{3\rho}p_{4\tau} = 2\left(-g^{\alpha\rho}g^{\beta\tau}+g^{\alpha\tau}g^{\beta\rho}\right)p_{1\alpha}p_{2\beta}p_{3\rho}p_{4\tau}\\
&= -2\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)+2\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right).
\end{aligned}$$

4. [15] Show that a particle with $E \gg m$ has approximate reciprocal velocity $1/v \approx 1 + m^2/2E^2$. If two neutrinos with different energies E_1 and E_2 arrive at the same time after travelling a distance *d*, find a formula for the difference in time Δt. In 1987, two neutrinos with the same mass and energies $E_1 = 6\text{ MeV}$ and $E_2 = 20\text{ MeV}$ arrived from SN1987a after travelling a distance $d = 160,000$ light years. Assuming they left at most $\Delta t < 10$ seconds apart, get an approximate limit on the mass *m* of the neutrinos.

Starting with eq. (2.8), we have

$$\frac{1}{\sqrt{1-v^2}} = \frac{E}{m},$$
$$1 - v^2 = m^2/E^2,$$
$$v^2 = 1 - m^2/E^2,$$
$$v^{-1} = \left(1 - m^2/E^2\right)^{-1/2} \approx 1 + \left(-\tfrac{1}{2}\right)\left(-m^2/E^2\right) = 1 + m^2/2E^2.$$

The amount of time it takes to travel a distance *d* is given by $t = d/v$, so the differences in times for traveling would be

$$\Delta t = d/v_1 - d/v_2 = d\left(1 + m^2/2E_1^2\right) - d\left(1 + m^2/2E_2^2\right) = \frac{dm^2}{2}\left(\frac{1}{E_1^2} - \frac{1}{E_2^2}\right).$$

The light traveled a distance $d = 160,000$ years (remember, we set *c* = 1), so we have

$$\frac{dm^2}{2}\left(\frac{1}{E_1^2} - \frac{1}{E_2^2}\right) < 10\text{ s},$$
$$\frac{\left(1.6\times10^5 \times 3.156\times10^7\text{ s}\right)m^2}{2}\left(\frac{1}{(6\text{ MeV})^2} - \frac{1}{(20\text{ MeV})^2}\right) < 10\text{ s},$$
$$\left(6.382\times10^{10}\text{ s/MeV}^2\right)m^2 < 10\text{ s},$$
$$m^2 < 1.57\times10^{-10}\text{ MeV}^2 = 157\text{ eV}^2,$$
$$m < 12.5\text{ eV}.$$

For reasons I can't recall, the actual limits was a bit better than this, but this was (at the time) one of the stronger limits on neutrinos masses.

6. **[10] Find a formula for *s* if particles of mass *m* and energy *E* collide with a stationary target of mass *m*. If you use *B* = 10 T magnets, how large in Earth radii would you have to make a proton collider to reach $\sqrt{s} = 8.00\text{ TeV}$?**

To make things simple, let's let the momentum of the moving particle be $p_1 = (E, 0, 0, p)$, and of the stationary one $p_2 = (m, 0, 0, 0)$. Then

$$s = (p_1 + p_2)^2 = (E+m)^2 - p^2 = E^2 + 2Em + m^2 - p^2 = m^2 + 2Em + m^2 = 2m(E+m).$$

If we want to reach $\sqrt{s} = 8.00\text{ TeV}$, we need to have

$$E + m = \frac{s}{2m} = \frac{(8000\text{ GeV})^2}{2(0.938\text{ GeV})} = 3.41\times10^7\text{ GeV} \approx E.$$

Because the protons are ultra-relativistic, we approximate $E = p$ (remember c = 1), and using eq. (1.5), we have

$$\left(\frac{R}{\text{km}}\right) = \left(\frac{p}{299.9\text{ GeV}}\right)\left(\frac{\text{T}}{B}\right) = \left(\frac{3.41\times10^7\text{ GeV}}{300\text{ GeV}}\right)\left(\frac{\text{T}}{10\text{ T}}\right) = (11{,}300\text{ km})\left(\frac{R_\oplus}{6{,}370\text{ km}}\right) = 1.79R_\oplus .$$

Hence the LHC can outperform (at least in terms of energy) a fixed target proton machine the size of the Earth. Now you know why we build colliders.