Solutions to Problems 10c

7. A muon neutrino with energy E = 10.0 MeV is attempting to scatter off of a stationary electron. What is s? What is the cross-section? How far would the neutrino have to travel on average before scattering if it were traveling through water?

The value of s is given by

$$s = E_{\text{tot}}^2 - p_{\text{tot}}^2 = (E+m)^2 - E^2 = (2E+m)m = (2\cdot 10.0\text{ MeV} + 0.511\text{ MeV})(0.511\text{ MeV})$$
$$= 10.48\text{ MeV}^2\,.$$

The cross-section can then be computed from eq. (10.49):

$$\sigma = \tfrac{1}{\pi} G_F^2 s\left[\tfrac{1}{4} - \sin^2\theta_W + \tfrac{4}{3}\sin^4\theta_W\right]$$
$$= \tfrac{1}{\pi}\left(1.166\times10^{-5}\text{ GeV}^{-2}\right)^2\left(10.48\times10^{-6}\text{ GeV}^2\right)\left[0.2500 - 0.2312 + \tfrac{4}{3}(0.2312)^2\right]$$
$$= 4.085\times10^{-17}\,\text{GeV}^{-2}\left(0.19732\text{ GeV}\cdot\text{fm}\right)^2 = 1.591\times10^{-18}\text{ fm}^2 = 1.591\times10^{-48}\text{ m}^2\,.$$

That's an awfully small cross-section! Now, the density of water is 1 g/cm^3. Water has a molecular weight of 18.015 g/mol, so this works out to 0.0555 mol/cm^3. Since there are ten electrons for each water molecule (8 from oxygen and one from each hydrogen), this comes out to 0.5551 mol electrons/cm^3, or

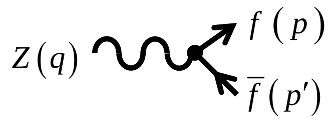
$$n_e = \left(0.5551\text{ mol/cm}^3\right)\left(6.022\times10^{23}\text{ /mol}\right) = 3.343\times10^{23}\text{ cm}^{-3} = 3.343\times10^{29}\text{ m}^{-3}\,.$$

We can then get the rate from $\Gamma = n\sigma(\Delta v)$. We are more interested in the mean free path, which is how far the neutrinos go before they scatter, which would be given by the velocity times the time, which would be $d = \Delta v/\Gamma = 1/(n\sigma)$, so we have

$$d = \frac{1}{n\sigma} = \frac{1}{\left(3.343\times10^{29}\text{ m}^{-3}\right)\left(1.591\times10^{-48}\text{ m}^2\right)} = 1.880\times10^{18}\text{m} = 199\,c\cdot\text{y}\,.$$

This is actually a considerable overestimate of the distance, because the most common scattering is off the nuclei, not the electrons, because of the much larger value of s. Nonetheless, you get the idea that the mean distance is large.

12. Determine the decay rates $\Gamma\left(Z \to f\,\bar{f}\right)$ for each possible final state fermion. Treat the fermions as massless, except for top, which is too heavy. Find the total decay rate for the *Z*, and the branching ratio to all neutrinos, any one lepton, and hadrons, and compare to the experimental values.



The relevant Feynman diagram for any of these decays is listed at right. The Feynman amplitude will depend on which fermion is in the final state, but always takes the form

$$i\mathcal{M} = \pm \frac{ie}{4\sin\theta_W \cos\theta_W}\varepsilon_\mu\left[\bar{u}'\gamma^\mu\left(a-\gamma_5\right)u\right]$$

The value of a is 1 for neutrinos, $1-4\sin^2\theta_W$ for charged leptons, $1-\frac{4}{3}\sin^2\theta_W$ for down quarks, and $1-\frac{8}{3}\sin^2\theta_W$ for up quarks. We multiply by the complex conjugate, and then sum and average over spins and polarizations

$$i\mathcal{M} = \mp \frac{ie}{4\sin\theta_W \cos\theta_W}\varepsilon_\nu^*\left[\bar{u}\left(a+\gamma_5\right)\gamma^\nu u'\right],$$

$$\left|i\mathcal{M}\right|^2 = \frac{e^2}{16\sin^2\theta_W\cos^2\theta_W}\varepsilon_\mu\varepsilon_\nu^*\left[\bar{u}'\gamma^\mu\left(a-\gamma_5\right)u\right]\left[\bar{u}\left(a+\gamma_5\right)\gamma^\nu u'\right],$$

$$\tfrac{1}{3}\sum\left|i\mathcal{M}\right|^2 = \frac{e^2}{48\sin^2\theta_W\cos^2\theta_W}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{M_Z^2}\right)\mathrm{Tr}\left[\not{p}'\gamma^\mu\left(a-\gamma_5\right)\not{p}\left(a+\gamma_5\right)\gamma^\nu\right]$$

$$= \frac{e^2}{48\sin^2\theta_W\cos^2\theta_W}\left\{-\mathrm{Tr}\left[\left(a-\gamma_5\right)^2\not{p}'\gamma^\mu\not{p}\gamma_\mu\right]+\frac{1}{M_Z^2}\mathrm{Tr}\left[\left(a-\gamma_5\right)^2\not{p}'\not{q}\not{p}\not{q}\right]\right\}$$

$$= \frac{e^2}{48\sin^2\theta_W\cos^2\theta_W}\mathrm{Tr}\left\{\left(a^2+1-2a\gamma_5\right)\left[2\not{p}'\not{p}+\frac{1}{M_Z^2}\not{p}'\left(\not{p}'+\not{p}\right)\not{p}\left(\not{p}'+\not{p}\right)\right]\right\}$$

$$= \frac{e^2}{48\sin^2\theta_W\cos^2\theta_W}8\left(a^2+1\right)\left(p\cdot p'\right) = \frac{e^2\left(a^2+1\right)}{12\sin^2\theta_W\cos^2\theta_W}\left(p+p'\right)^2 = \frac{e^2M_Z^2\left(a^2+1\right)}{12\sin^2\theta_W\cos^2\theta_W}.$$

We now proceed to the decay rate in the usual way.

$$\Gamma = \frac{D}{2M_Z} = \frac{1}{2M_z}\frac{p}{16\pi^2 M_Z}\int\tfrac{1}{3}\sum\left|i\mathcal{M}\right|^2 d\Omega = \frac{4\pi}{64\pi^2 M_Z}\frac{e^2M_Z^2\left(a^2+1\right)}{12\sin^2\theta_W\cos^2\theta_W} = \frac{\alpha M_Z\left(a^2+1\right)}{48\sin^2\theta_W\cos^2\theta_W}.$$

We need to remember to multiply by three for any quark final states (to account for colors), and then substitute in the appropriate value for a. We substitute in $\alpha = \frac{1}{128}$, appropriate at the Z mass, as well as $\sin^2\theta_W = 0.2312$, $\cos^2\theta_W = 0.7688$ and $M_Z = 91.19\text{ GeV}$, to find

$$\Gamma\left(Z \to \bar{\ell}\ell\right) = \frac{\alpha M_Z\left[1+\left(1-4\sin^2\theta_W\right)^2\right]}{48\sin^2\theta_W\cos^2\theta_W} = \frac{\left(91.19\text{ GeV}\right)\left[1+\left(1-4\times 0.2312\right)^2\right]}{128\times 48\times 0.2312\times 0.7688} = 0.0840\text{ GeV},$$

$$\Gamma\left(Z \to \bar{\nu}_i\nu_i\right) = \frac{\alpha M_Z}{24\sin^2\theta_W\cos^2\theta_W} = \frac{\left(91.19\text{ GeV}\right)}{128\times 24\times 0.2312\times 0.7688} = 0.1670\text{ GeV},$$

$$\Gamma\left(Z \to \bar{u}_i u_i\right) = \frac{\alpha M_Z\left[1+\left(1-\tfrac{8}{3}\sin^2\theta_W\right)^2\right]}{16\sin^2\theta_W\cos^2\theta_W} = \frac{\left(91.19\text{ GeV}\right)\left[1+\left(1-\tfrac{8}{3}\times 0.2312\right)^2\right]}{128\times 16\times 0.2312\times 0.7688} = 0.2873\text{ GeV},$$

$$\Gamma\left(Z \to \bar{d}_i d_i\right) = \frac{\alpha M_Z\left[1+\left(1-\tfrac{4}{3}\sin^2\theta_W\right)^2\right]}{48\sin^2\theta_W\cos^2\theta_W} = \frac{\left(91.19\text{ GeV}\right)\left[1+\left(1-\tfrac{4}{3}\times 0.2312\right)^2\right]}{128\times 16\times 0.2312\times 0.7688} = 0.3704\text{ GeV}.$$

The total decay rate is the sum of all these, summed over three charged leptons, three neutrinos, two up-type quarks (top is too heavy) and three down-type quarks, for a total of

$$\begin{aligned}\Gamma(Z) &= 3\Gamma\left(Z \to \bar{\ell}\ell\right) + 3\Gamma\left(Z \to \bar{\nu}_i\nu_i\right) + 2\Gamma\left(Z \to \bar{u}_i u_i\right) + 3\Gamma\left(Z \to \bar{d}_i d_i\right) \\ &= 3\left(0.0840\text{ GeV}\right) + 3\left(0.1670\text{ GeV}\right) + 2\left(0.2873\text{ GeV}\right) + 3\left(0.3704\text{ GeV}\right) = 2.439\,\text{GeV}.\end{aligned}$$

This should be compared to the experimental value of $\Gamma = 2.495\,\text{GeV}$, about a 2% error. The relevant branching ratios are:

$$BR\left(Z \to \bar{\ell}\ell\right) = \frac{\Gamma\left(Z \to \bar{\ell}\ell\right)}{\Gamma(Z)} = \frac{0.0840\text{ GeV}}{2.439\text{ GeV}} = 3.44\%,$$

$$BR\left(Z \to \bar{\nu}\nu\right) = \frac{3\Gamma\left(Z \to \bar{\nu}_i\nu_i\right)}{\Gamma(Z)} = \frac{3\times 0.1670\text{ GeV}}{2.439\text{ GeV}} = 20.54\%,$$

$$\begin{aligned}BR\left(Z \to \text{hadrons}\right) &= \frac{2\Gamma\left(Z \to \bar{u}_i u_i\right) + 3\Gamma\left(Z \to \bar{d}_i d_i\right)}{\Gamma(Z)} = \frac{2\times 0.2873\text{ GeV} + 3\times 0.3704\text{ GeV}}{2.439\text{ GeV}} \\ &= 69.12\%.\end{aligned}$$

The actual branching fractions are about 3.37% for the leptons, about 20.0% for invisible, and about 69.9% for hadrons. A closer examination will show that we got the partial decay to leptons and neutrinos almost exactly right, but we underestimated the hadronic cross section, probably due to strong corrections.