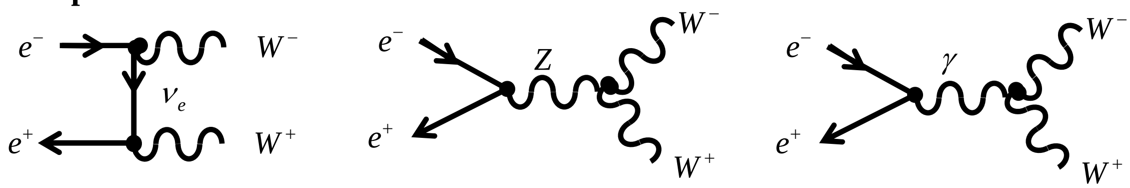
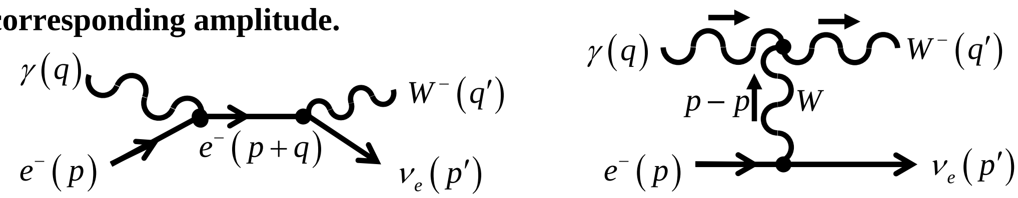
Solutions to Problems 10b

4. Draw all tree-level Feynman diagrams for the process $e^+e^- \to W^+W^-$. You do not need to compute them.



5. Draw all tree-level Feynman diagrams for the process $e^-\gamma \to W^-\nu_e$, and write the corresponding amplitude.



To calculate the Feynman amplitude, it is necessary that we keep track of which direction the momentum is flowing in the diagram on the right. The rule for this diagram assumes that all momenta are flowing into the triple gauge boson vertex, so we will treat the outgoing W^- boson as having a momentum of $-q'$. We therefore have

$$i\mathcal{M} = (ie)\left(-\frac{ig}{2\sqrt{2}}\right)\varepsilon_\nu'^*\varepsilon_\mu \left\{ \begin{array}{c} \bar{u}'\gamma^\nu(1-\gamma_5)\dfrac{i(\not{p}+\not{q}+m)}{(p+q)^2-m^2}\gamma^\mu u + \\ \left[\bar{u}'\gamma^\alpha(1-\gamma_5)u\right]\dfrac{i\left[-g_{\alpha\beta}+(p-p')_\alpha(p-p')_\beta/M_W^2\right]}{(p-p')^2-M_W^2} \\ \times\left[(-q'-q)^\beta g^{\nu\mu}+(q+p'-p)^\nu g^{\beta\mu}+(p-p'+q')^\mu g^{\beta\nu}\right] \end{array} \right\}$$

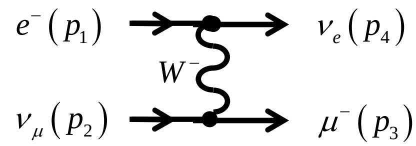
As you can imagine, we don't actually want to do anything with this mess. The electron mass *m* can almost certainly be safely neglected, and you can take advantage of the fact that $q\cdot\varepsilon = q'\cdot\varepsilon' = 0$ to simplify it somewhat. The complicated terms from the propagator can be eliminated by using the fact that

$$\bar{u}'\gamma^\alpha(1-\gamma_5)u(p-p')_\alpha = \bar{u}'(\not{p}-\not{p}')(1-\gamma_5)u = \bar{u}'\not{p}(1-\gamma_5)u = \bar{u}'(1+\gamma_5)\not{p}u = 0.$$

We then can simplify things a lot to

$$i\mathcal{M} = \frac{ieg}{2\sqrt{2}}\bar{u}'(1+\gamma_5)\left\{\not{\varepsilon}'^*\frac{\not{p}+\not{q}}{2p\cdot q}\not{\varepsilon}+\frac{1}{2q\cdot q'}\left[(\not{q}'+\not{q})(\varepsilon'^*\cdot\varepsilon)-2(q\cdot\varepsilon'^*)\not{\varepsilon}-2(q'\cdot\varepsilon)\not{\varepsilon}'^*\right]\right\}u$$

8. Find the amplitude for $e^-\nu_\mu \to \mu^-\nu_e$. Assume the energies are well below the *W*-mass, so you can ignore the momentum in the denominator of the *W*-propagator. Treat all four leptons as massless. Write your answer exclusively in terms of G_F and *s*. Find the differential and total cross section for this scattering.



There is only one diagram, sketched at right. The amplitude is given by

$$i\mathcal{M}=\left(-\frac{ig}{2\sqrt{2}}\right)^2\left[\bar{u}_3\gamma^\mu(1-\gamma_5)u_2\right]\left[\bar{u}_4\gamma^\nu(1-\gamma_5)u_1\right]\frac{i\left[-g_{\mu\nu}+(p_1-p_4)_\mu(p_1-p_4)_\nu/M_W^2\right]}{(p_1-p_4)^2-M_W^2}$$

$$\approx\frac{-ig^2}{8M_W^2}\left[\bar{u}_3\gamma^\mu(1-\gamma_5)u_2\right]\left[\bar{u}_4\gamma_\mu(1-\gamma_5)u_1\right]=-i\tfrac{1}{\sqrt{2}}G_F\left[\bar{u}_3\gamma^\mu(1-\gamma_5)u_2\right]\left[\bar{u}_4\gamma_\mu(1-\gamma_5)u_1\right].$$

As usual, we find the complex conjugate,

$$(i\mathcal{M})^*=i\tfrac{1}{\sqrt{2}}G_F\left[\bar{u}_2(1+\gamma_5)\gamma^\nu u_3\right]\left[\bar{u}_1(1+\gamma_5)\gamma_\nu u_4\right],$$

then multiply by it and sum or average over spins. There is a complication with the initial neutrino, however. Because it is always left-handed, we should not average over its spin. On the other hand, if we sum over it, only the left-handed part can contribute. Hence we sum over the initial spin of the neutrino, and only average over the electron spin. We therefore have

$$\begin{aligned}
\tfrac{1}{2}\sum\left|i\mathcal{M}\right|^2&=\tfrac{1}{4}G_F^2\operatorname{Tr}\left[\not{p}_3\gamma^\mu(1-\gamma_5)\not{p}_2(1+\gamma_5)\gamma^\nu\right]\operatorname{Tr}\left[\not{p}_4\gamma_\mu(1-\gamma_5)\not{p}_1(1+\gamma_5)\gamma_\nu\right]\\
&=G_F^2\operatorname{Tr}\left[(1-\gamma_5)\not{p}_3\gamma^\mu\not{p}_2\gamma^\nu\right]\operatorname{Tr}\left[(1-\gamma_5)\not{p}_4\gamma_\mu\not{p}_1\gamma_\nu\right]\\
&=16G_F^2\left[p_3^\mu p_2^\nu+p_2^\mu p_3^\nu-(p_2\cdot p_3)g^{\mu\nu}-i\varepsilon^{\alpha\mu\beta\nu}p_{3\alpha}p_{2\beta}\right]\\
&\quad\times\left[p_{4\mu}p_{1\nu}+p_{4\nu}p_{1\mu}-(p_1\cdot p_4)g_{\mu\nu}-i\varepsilon^{\rho\ \ \sigma}_{\ \mu\ \nu}p_{4\rho}p_{1\sigma}\right]\\
&=16G_F^2\left\{\begin{array}{l}(p_3\cdot p_4)(p_1\cdot p_2)(1+1)+(p_3\cdot p_1)(p_2\cdot p_4)(1+1)+(p_2\cdot p_3)(p_1\cdot p_4)\\ \times(-1-1-1-1+4)+2\left[(p_3\cdot p_4)(p_1\cdot p_2)-(p_3\cdot p_1)(p_2\cdot p_4)\right]\end{array}\right\}\\
&=64G_F^2(p_3\cdot p_4)(p_1\cdot p_2).
\end{aligned}$$

It isn't hard to see that $s=(p_1+p_2)^2=2p_1\cdot p_2$ and $s=(p_3+p_4)^2=2p_3\cdot p_4$, so that this can be simplified to $\tfrac{1}{2}\sum\left|i\mathcal{M}\right|^2=16G_F^2s^2$.

We now proceed towards the cross-section in the usual way. We have

$$\sigma=\frac{D}{4\left|E_2\mathbf{p}_1-E_1\mathbf{p}_2\right|}=\frac{D}{8E^2}=\frac{1}{2s}\cdot\frac{\left|\mathbf{p}_3\right|}{16\pi^2E_{\text{cm}}}\int\tfrac{1}{2}\sum\left|i\mathcal{M}\right|^2d\Omega=\frac{1}{64\pi^2s}16G_F^2s^2\int d\Omega=\frac{1}{4\pi^2}G_F^2s\int d\Omega.$$

The differential and total cross-section is, of course, just

$$\frac{d\sigma}{d\Omega}=\frac{1}{4\pi^2}G_F^2s,\quad\sigma=\frac{1}{\pi}G_F^2s.$$