**Solutions to Midterm Exam** Name: ____________________

Particle Physics, Fall 2012 October 16, 2012

Part I: Short Answer [50 points]

**For each of the following, give a short answer (2-3 sentences, or a formula). [5 points each]**

1. **Explain qualitatively (a) how we accelerate particles to high energy, and (b) how we get them to go in a circle. Equations are nice but not necessary.**

Electromagnetic forces are used to accelerate and guide particles, using the Lorentz force equation $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Electric field accelerate them, increasing their energy, while magnetic fields guide them in a circle.

2. **Suppose I were practicing archery, and I was trying to hit spherical water balloons of radius *R* with small arrows. What would be the cross-section for these targets?**

The cross-section is the area as viewed from one direction. Since a sphere has the silhouette of a circle, it will have cross-section area of $\pi R^2$

3. **Which of the following equations are manifestly Lorentz covariant?**
   **(a)** $p_\mu \varepsilon^\mu p^\nu = p_\alpha p^\alpha \varepsilon^\nu$ **- Yes**

   **(b)** $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$ **- No, indices can't be repeated down**

   **(c)** $p^\mu q_\mu = p_\mu q^\mu$ **- Yes**

   **(d)** $p^\mu = m$ **- Unmatched index**

   **(e)** $\varepsilon_{\mu\nu\alpha\beta} = -\varepsilon^{\mu\nu\alpha\beta}$ **- Can't match up with down**

4. **Explain physically or using equations what it means if your theory respects parity**

If parity is respected, the laws of physics will look the same if you see it in a mirror. Mathematically, the easiest way to impose parity is to let $\mathbf{x} \to -\mathbf{x}$.

5. **Explain briefly, according to the Dirac theory, why it is that positive energy electrons do not normally "decay" and become negative energy electrons.**

According to the Dirac theory, all these negative energy states are already filled, and therefore the electron can't "fall down" to such a state, by the Pauli exclusion principle.

6. **The unstable *Z*-boson is listed as having a mass of 91.19 GeV. This implies that for a *Z* at rest, $H|Z,\mathbf{p}=0\rangle = E|Z,\mathbf{p}=0\rangle$ with $E = 91.19\text{ GeV}$. Explain why we know for sure that, in fact, the *Z*-boson is not an eigenstate of the full Hamiltonian.**

An exact eigenstate evolves according to $|\Psi(t)\rangle = e^{-iEt}|\Psi(0)\rangle$. If the *Z*-boson were an exact eigenstate of the full Hamiltonian, then it would never decay, but the problem says that it is unstable.

7. **In weak interactions, you can get non-zero matrix elements of the form $\langle 0|\mathcal{H}|W^-,u,\bar{q}\rangle$, where $W^-$ is the *W*-boson with charge –1, u is the up quark with charge +2/3, and $\bar{q}$ is an anti-quark of some sort. Tell me the charge of $\bar{q}$, and the charge of the corresponding quark *q*.**

Charge must be conserved, so the sum of the charges on the right must add to zero. If we make *x* the charge of the unknown anti-quark, then $0 = -1 + \frac{2}{3} + x$, so $x = +\frac{1}{3}$. The corresponding quark will have charge $-\frac{1}{3}$.

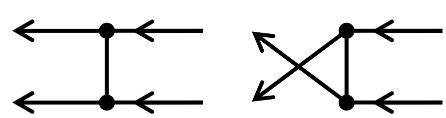
8. **Suppose you are measuring the cross-section for some process, and you get very close to the mass of an intermediate particle (resonance). What happens, qualitatively, to the cross-section?**

When you get close to resonance, there will be a very large increase in the cross-section.

9. **When you have a resonance, how can you tell from a graph of cross-section vs. energy what the decay rate or width $\Gamma$ is for that intermediate particle?**

The width $\Gamma$ is approximately the full width at half maximum of the resonance. That is, find the highest point in the cross section, find the places where the curve is half the peak, and take the difference in energies of those two places. That is (approximately) $\Gamma$.

10. **In the Feynman diagrams at right, the arrow is a fermion, and the solid line is a boson. Would you add or subtract the contributions to the Feynman amplitude, and why?**



Because the two diagrams differ by the exchange of an external fermion line, you would subtract them.

**Part II: Calculation [150 points]**
Each problem has its corresponding point value marked. Solve the equations on separate paper.

**11. [15] According to the particle data book, the K$^+$ meson has a mass of 493.7 MeV and a mean lifetime of** $\tau = 1.238\times10^{-8}$ s

**(a) If a K$^+$ meson had an energy of *E* = 2350 MeV, how long would it last and how far would it go if it lasts one average lifetime?**

We start with the equation

$$\gamma = \frac{E}{m} = \frac{t}{\tau} = \frac{1}{\sqrt{1-v^2}}.$$

We first see that

$$\gamma = \frac{E}{m} = \frac{2350\text{ MeV}}{493.7\text{ MeV}} = 4.76\,.$$

We then get the amount of time it actually lasts as

$$t = \gamma\tau = 4.76\times1.238\times10^{-8}\text{ s} = 5.89\times10^{-8}\text{ s}\,.$$

We will need the velocity, which we can find pretty readily from

$$\gamma = \frac{1}{\sqrt{1-v^2}}\,,$$

$$1-v^2 = \frac{1}{\gamma^2}\,,$$

$$v = \sqrt{1-\gamma^{-2}} = \sqrt{1-4.76^{-2}} = 0.9777\,.$$

Distance traveled is then velocity times time, which works out to

$$d = vt = 0.9777\left(5.89\times10^{-8}\text{ s}\right)\left(2.998\times10^{8}\text{ m/s}\right) = 17.3\text{ m}\,.$$

**(b) What is the width $\Gamma$ of a K$^+$, in eV?**

The width is given by

$$\Gamma = \frac{1}{\tau} = \frac{6.582\times10^{-16}\text{ eV}\cdot\text{s}}{1.238\times10^{-8}\text{ s}} = 5.317\times10^{-8}\text{ eV}\,.$$

**(c) The branching ratio for $K^+\to\pi^+\pi^0$ is 20.7%. What is $\Gamma\left(K^+\to\pi^+\pi^0\right)$, in eV?**

The partial width can then be computed from

$$\Gamma\left(K^+\to\pi^+\pi^0\right) = BR\left(K^+\to\pi^+\pi^0\right)\Gamma\left(K^+\right) = \left(0.207\right)\left(5.317\times10^{-8}\text{ eV}\right) = 1.100\times10^{-8}\text{ eV}\,.$$

**12. [15] The differential cross-section for $e^+e^- \to \mu^+\mu^-$ at high energies is given by**

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\left(1+\cos^2\theta\right)$$

**(a) Calculate the total cross-section**

We simply integrate this over angles to yield

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{\alpha^2}{16E^2}\int_0^{2\pi} d\phi \int_{-1}^{1}\left(1+\cos^2\theta\right) d\cos\theta = \frac{2\pi\alpha^2}{16E^2}\left(\cos\theta + \tfrac{1}{3}\cos^2\theta\right)\Big|_{\cos\theta=-1}^{\cos\theta=1}$$
$$= \frac{\pi\alpha^2}{8E^2}\left(1+\tfrac{1}{3}-\left(-1\right)-\left(-\tfrac{1}{3}\right)\right) = \frac{\pi\alpha^2}{8E^2}\cdot\frac{8}{3} = \frac{\pi\alpha^2}{3E^2}.$$

**(b) Convert to barns if E = 10.0 GeV. Use $\alpha = 1/137$.**

We have

$$\sigma = \frac{\pi\left(0.197\ \text{GeV}\cdot\text{fm}\right)^2}{3\cdot 137^2\left(10.0\ \text{GeV}\right)^2}\cdot\frac{\text{b}}{100\ \text{fm}^2} = 2.165\times 10^{-10}\,\text{b} = 0.2165\ \text{nb}.$$

**(c) If an $e^+e^-$ collider is operating with each beam at *E* = 10.0 GeV at a luminosity of $L = 5.67\ \mu\text{b}^{-1}\text{s}^{-1}$, how many $\mu^+\mu^-$ pairs will it make in one day?**

The number produced is

$$N\left(\mu^+\mu^-\right) = \sigma\int L dt = \left(2.165\times 10^{-10}\ \text{b}\right)\left(5.67\ \mu\text{b}^{-1}\text{s}^{-1}\right)\left(10^6\ \mu\text{b/b}\right)\left(24\ \text{h}\right)\left(60\ \text{m/h}\right)\left(60\ \text{s/m}\right) = 106.$$

**13. [10] A collision process takes the form $A\left(p_1\right)B\left(p_2\right)\to C\left(p_3\right)D\left(p_4\right)$, where the masses of the four particles are $m_1$, $m_2$, $m_3$, and $m_4$ respectively. Show that $p_2\cdot p_4$ can be written in terms of $p_1\cdot p_3$.**

By conservation of four-momentum, we know that $p_1 + p_2 = p_3 + p_4$. Rearranging this slightly, we have $p_1 - p_3 = p_4 - p_2$. Squaring this expression, we have

$$\left(p_1 - p_3\right)^2 = \left(p_4 - p_2\right)^2,$$
$$p_1^2 + p_3^2 - 2p_1\cdot p_3 = p_2^2 + p_4^2 - 2p_2\cdot p_4,$$
$$m_1^2 + m_3^2 - 2p_1\cdot p_2 = m_2^2 + m_4^2 - 2p_2\cdot p_4,$$
$$2p_2\cdot p_4 + m_1^2 + m_3^2 = 2p_1\cdot p_2 + m_2^2 + m_4^2,$$
$$p_2\cdot p_4 = p_1\cdot p_2 + \tfrac{1}{2}\left(m_2^2 + m_4^2 - m_1^2 - m_3^2\right).$$

**14. [20] Consider the process $\psi(p_1)\psi^*(p_2)\to\phi(k_1)\phi(k_2)$ in the center of mass frame. Let the mass of the $\psi$'s be *m* and the mass of the $\phi$'s be *M*.**

**(a) Assume the initial particles have energy *E* in the center of mass frame. Tell me the energy of the final particles, and the magnitude of the momentum of the initial and final particles. You must give arguments for your answers, not just the answers.**

Since we are in the center of mass frame, the momenta will be equal and opposite. Since the initial particles have equal momenta and equal masses, they will have equal energies *E*. The total energy 2*E* will be divided between the two final particles. Since the final particles have matching momenta and matching masses, they will also have equal energies. Hence each of them gets energy *E*.

The momenta can be found by the fact that $E^2=\mathbf{p}^2+m^2$, so we have

$$p=|\mathbf{p}_1|=|\mathbf{p}_2|=\sqrt{E^2-m^2}\ ,\quad k=|\mathbf{k}_1|=|\mathbf{k}_2|=\sqrt{E^2-M^2}\ .$$

**(b) Write out explicitly all four components of all four momenta. You may assume the initial particles are coming in along the $\pm x^3$ axes. The final particles will go in an arbitrary direction.**

We simply include factors corresponding to the various directions, and write

$$p_1=(E,0,0,p)\,,\quad k_1=(E,k\sin\theta\cos\phi,k\sin\theta\sin\phi,k\cos\theta),$$
$$p_2=(E,0,0,-p)\,,\quad k_2=(E,-k\sin\theta\cos\phi,-k\sin\theta\sin\phi,-k\cos\theta).$$

**(c) Calculate all six dot-products of initial and final momenta explicitly, *i.e.*, tell me**

$$p_1\cdot p_2\,,\quad k_1\cdot k_2\,,\quad p_1\cdot k_1\,,\quad p_1\cdot k_2\,,\quad p_2\cdot k_1\,,\quad p_2\cdot k_2\,.$$

These are straightforward:

$$p_1\cdot p_2=E^2+p^2\ ,$$
$$k_1\cdot k_2=E^2+k^2\sin^2\theta\cos^2\phi+k^2\sin^2\theta\sin^2\phi+k^2\cos^2\theta=E^2+k^2\sin^2\theta+k^2\cos^2\theta$$
$$=E^2+k^2\ ,$$
$$p_1\cdot k_1=E^2-kp\cos\theta\,,$$
$$p_1\cdot k_2=E^2+kp\cos\theta\,,$$
$$p_2\cdot k_1=E^2+kp\cos\theta\,,$$
$$p_2\cdot k_2=E^2-kp\cos\theta\,.$$

**15. [15] A high energy anti-neutrino $(m_\nu = 0)$ with energy *E* collides with an electron at rest ($m_e = 0.511\text{ MeV}$).**

**(a) Find a simple formula for the center of mass energy squared *s* in terms of *E* and $m_e$.**

If we let $p_\nu$ and $p_e$ be the two momenta, then we have

$$s = \left(p_\nu + p_e\right)^2 = p_\nu^2 + p_e^2 + 2p_\nu \cdot p_e = 0 + m_e^2 + 2m_e E\,.$$

We found the dot product by using the fact that

$$p_\nu = \left(E, 0, 0, E\right), \quad p_e = \left(m_e, 0, 0, 0\right), \quad p_\nu \cdot p_e = m_e E\,.$$

**(b) How big must *E* be to produce a *W* boson ($m_W = 80.40\text{ GeV}$) via the process $\bar{\nu}_e e^- \to W^-$?**

For the process, we need $s = p_W^2 = M_W^2$. Solving for the energy, we have

$$2m_e E = s - m_e^2\,,$$

$$E = \frac{s - m_e^2}{2m_e} = \frac{M_W^2 - m_e^2}{2m_e} = \frac{\left(80.40\text{ GeV}\right)^2 - \left(0.000511\text{ GeV}\right)^2}{2\left(0.000511\text{ GeV}\right)}$$

$$= 6.325 \times 10^6\text{ GeV} = 6.325\text{ PeV}\,.$$

It's not that often we get to use PeV.

**16. [25] Consider a renormalizable theory with two charged spin 0 particles, $\psi_1$ with charge +1, and $\psi_3$ with charge +3. They are not equivalent to their anti-particles $\psi_1^*$ and $\psi_3^*$.**

**(a) Write down all possible renormalizable matrix elements of the form $\langle 0|\mathcal{H}|X\rangle$, where *X* has more than two particles, and figure out which ones must be real.**

To be renormalizable, there must be no more than four particles. The total charge must be zero. One way to do this is to have exactly equal numbers of $\psi_1$ with $\psi_1^*$, and $\psi_3$ with $\psi_3^*$, which yields the following three matrix elements:

$$\langle 0|\mathcal{H}|\psi_1\psi_1\psi_1^*\psi_1^*\rangle = \lambda_1\,, \quad \langle 0|\mathcal{H}|\psi_1\psi_3\psi_1^*\psi_3^*\rangle = \lambda_2\,, \quad \langle 0|\mathcal{H}|\psi_3\psi_3\psi_3^*\psi_3^*\rangle = \lambda_3\,.$$

We can use the Hermitian and anti-particle property to show these are real:

$$\lambda_1^* = \langle 0|\mathcal{H}|\psi_1\psi_1\psi_1^*\psi_1^*\rangle^* = \langle \psi_1\psi_1\psi_1^*\psi_1^*|\mathcal{H}|0\rangle = \langle 0|\mathcal{H}|\psi_1^*\psi_1^*\psi_1\psi_1\rangle = \lambda_1 \,,$$
$$\lambda_2^* = \langle 0|\mathcal{H}|\psi_1\psi_3\psi_1^*\psi_3^*\rangle^* = \langle \psi_1\psi_3\psi_1^*\psi_3^*|\mathcal{H}|0\rangle = \langle 0|\mathcal{H}|\psi_1^*\psi_3^*\psi_1\psi_3\rangle = \lambda_2 \,,$$
$$\lambda_3^* = \langle 0|\mathcal{H}|\psi_3\psi_3\psi_3^*\psi_3^*\rangle^* = \langle \psi_3\psi_3\psi_3^*\psi_3^*|\mathcal{H}|0\rangle = \langle 0|\mathcal{H}|\psi_3^*\psi_3^*\psi_3\psi_3\rangle = \lambda_3 \,.$$

We can also try to balance charge by having, say, one extra $\psi_3$ and cancelling it with a bunch of $\psi_1^*$'s, or we can do the same thing with an extra $\psi_3^*$ and cancelling it with $\psi_1$'s. The resulting matrix elements we will name as

$$\langle 0|\mathcal{H}|\psi_3\psi_1^*\psi_1^*\psi_1^*\rangle = g \,, \quad \langle 0|\mathcal{H}|\psi_1\psi_1\psi_1\psi_3^*\rangle = h.$$

These matrix elements are complex conjugates of each other:

$$g^* = \langle 0|\mathcal{H}|\psi_3\psi_1^*\psi_1^*\psi_1^*\rangle^* = \langle \psi_3\psi_1^*\psi_1^*\psi_1^*|\mathcal{H}|0\rangle = \langle 0|\mathcal{H}|\psi_3^*\psi_1\psi_1\psi_1\rangle = h.$$

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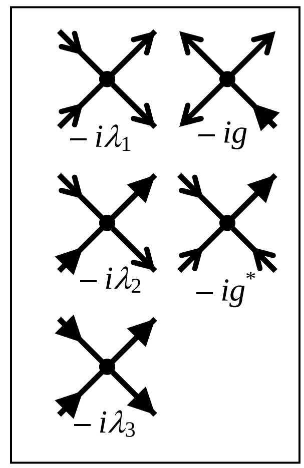
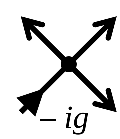
**(b) Make up a diagrammatic notation for the particles $\psi_1$ and $\psi_3$. Draw all possible vertices for this theory, and give me the corresponding factor that should be included for this theory.**




A normal notation would be a single arrow for $\psi_1$ and a triple arrow for $\psi_3$, but that's hard to draw, so I'll use an open arrow for $\psi_1$ and a closed arrow for $\psi_3$. There are five matrix elements, and hence five possible vertices in this theory, as sketched at right.

**(c) Consider the scattering $\psi_3\psi_1^* \to \psi_1\psi_1$. Draw the relevant Feynman diagram, and give me the relevant Feynman amplitude. Then find the differential and total cross-section, treating all particles as massless, if the energy of each of the initial particles is *E*.**




There is only one tree-level diagram, sketched at right. The Feynman amplitude is $i\mathcal{M} = -ig$. We would then proceed to the cross section in the usual way

$$\sigma = \frac{D}{4|E_p\mathbf{k} - E_k\mathbf{p}|} = \frac{1}{8E^2}\frac{|\mathbf{p}_2|}{16\pi^2 E_{cm}}\int |i\mathcal{M}|^2\, d\Omega = \frac{1}{8E^2}\frac{E|-ig|^2}{16\pi^2(2E)}\int d\Omega = \frac{|g|^2}{256\pi^2E^2}\int d\Omega \,,$$

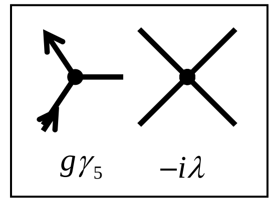
$$\frac{d\sigma}{d\Omega} = \frac{|g|^2}{256\pi^2E^2}\,.$$

We now integrate over angles, but noting that there are identical particles in the final state, we need to throw in a factor of ½ to avoid double counting, so we have

$$\sigma = \frac{|g|^2}{128\pi E^2}\,.$$

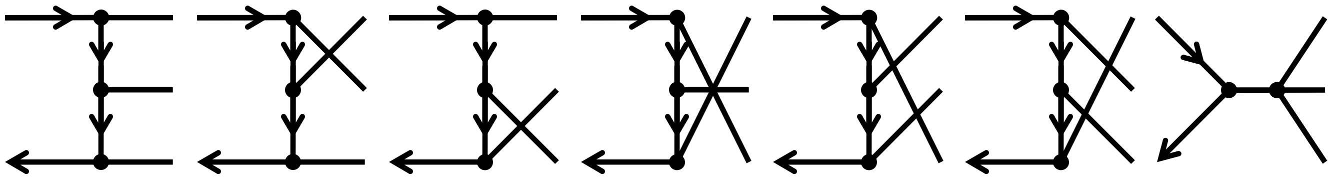
**17. [25] Working in the full $\bar{\psi}\psi\phi$ theory with pseudoscalar couplings, sketch all seven tree-level diagrams for the process**

$$\psi(p_1)\bar{\psi}(p_2) \to \phi(k_1)\phi(k_2)\phi(k_3)$$




**Then, carefully write the Feynman invariant amplitude for two of them: one which does involve the $\lambda$ coupling, and one which does not involve it. You don't have to do the other diagrams, nor do you have to do anything with the resulting amplitudes. The Feynman rules for the only two allowed vertices are given above.**

The seven relevant diagrams are sketched below



Only the last one involves the $\lambda$ coupling. For the other diagrams, the upper fermion propagator has momentum $p_1 - k_i$, where $k_i$ is the momentum of the $\phi$ attached to the upper vertex, and the lower fermion propagator has momentum $k_j - p_2$. The last diagram has a scalar propagator with momentum $p_1 + p_2$. Putting it all together, we find the Feynman amplitude is

$$i\mathcal{M} = i^2 g^3 \sum_{i,j} \frac{\bar{v}_2\gamma_5\left(\not{k}_j - \not{p}_2 + m\right)\gamma_5\left(\not{p}_1 - \not{k}_i + m\right)\gamma_5 u_1}{\left[\left(p_1 - k_i\right)^2 - m^2\right]\left[\left(k_j - p_2\right)^2 - m^2\right]} + \frac{ig\left(-i\lambda\right)\bar{v}_2\gamma_5 u_1}{\left(p_1 + p_2\right)^2 - M^2}$$

The sum over $i$ and $j$ is over all possible pairs chosen from $\{1,2,3\}$, but picking $i \neq j$, so six terms in all.

**18. [25] It is possible (though not likely) that one of the decays of the top quark will be**

$$t(p_1, s_1) \to b(p_2, s_2)h^+(p_3),$$

**where *t* and *b* are the top and bottom quark (both fermions) and $h^+$ is a charged scalar. Assume the amplitude for this process takes the form**

$$i\mathcal{M} = \bar{u}_2\left(-i\alpha + \beta\gamma_5\right)u_1 \,,$$

**where $\alpha$ and $\beta$ are real constants. Calculate the decay rate $\Gamma\left(t \to bh^+\right)$. Assume the top quark has mass $m_t$, the bottom quark has mass 0, and the scalar field has mass $m_h$.**

The amplitude can't be simplified, so we start by finding

$$\left(i\mathcal{M}\right)^* = \bar{u}_1\left(i\alpha - \beta\gamma_5\right)u_2 \,.$$

So the square of the amplitude is

$$\left|i\mathcal{M}\right|^2 = \bar{u}_1\left(i\alpha - \beta\gamma_5\right)u_2\bar{u}_2\left(-i\alpha + \beta\gamma_5\right)u_1 = \mathrm{Tr}\left[u_1\bar{u}_1\left(i\alpha - \beta\gamma_5\right)u_2\bar{u}_2\left(-i\alpha + \beta\gamma_5\right)\right]$$

Now, the initial top quark has random spin, so we average over this spin. For the bottom quark, we would sum over spins. So we are actually interested in

$$\tfrac{1}{2}\sum_{s_1,s_2}\left|i\mathcal{M}\right|^2 = \tfrac{1}{2}\mathrm{Tr}\left[\left(\not{p}_1 + m_t\right)\left(i\alpha - \beta\gamma_5\right)\not{p}_2\left(-i\alpha + \beta\gamma_5\right)\right]$$

Keeping in mind that $\not{p}_2$ anti-commutes with $\gamma_5$, this can be rewritten as

$$\begin{aligned}\tfrac{1}{2}\sum_{s_1,s_2}\left|i\mathcal{M}\right|^2 &= \tfrac{1}{2}\mathrm{Tr}\left[\left(\not{p}_1 + m_t\right)\not{p}_2\left(i\alpha + \beta\gamma_5\right)\left(-i\alpha + \beta\gamma_5\right)\right] = \tfrac{1}{2}\mathrm{Tr}\left[\left(\not{p}_1 + m_t\right)\not{p}_2\left(\alpha^2 + \beta^2\gamma_5^2\right)\right]\\ &= \tfrac{1}{2}\left(\alpha^2 + \beta^2\right)\mathrm{Tr}\left(\not{p}_1\not{p}_2 + m_t\not{p}_2\right) = 2\left(\alpha^2 + \beta^2\right)\left(p_1\cdot p_2 + 0\right) = 2\left(\alpha^2 + \beta^2\right)p_1\cdot p_2\,.\end{aligned}$$

Now, from conservation of four-momentum, we know that $p_1 = p_2 + p_3$. Rearranging this slightly, we have $p_3 = p_1 - p_2$. Squaring this, we have

$$\begin{aligned}p_3^2 &= \left(p_1 - p_2\right)^2 = p_1^2 + p_2^2 - 2p_1\cdot p_2\,,\\ m_h^2 &= m_t^2 + 0 - 2p_1\cdot p_2\,,\\ 2p_1\cdot p_2 &= m_t^2 - m_h^2\,.\end{aligned}$$

Substituting this into our previous expression, we have

$$\tfrac{1}{2}\sum_{s_1,s_2}\left|i\mathcal{M}\right|^2 = \left(\alpha^2 + \beta^2\right)\left(m_t^2 - m_h^2\right).$$

We now proceed to the decay rate, which is given by

$$\Gamma = \frac{D}{2m_t} = \frac{1}{2m_t}\frac{p}{16\pi^2 E_{\mathrm{cm}}}\int\tfrac{1}{2}\sum_{s_1,s_2}\left|i\mathcal{M}\right|^2 d\Omega = \frac{p}{32\pi^2 m_t^2}\left(\alpha^2 + \beta^2\right)\left(m_t^2 - m_h^2\right)\int d\Omega\,.$$

There are no complications involving the final state particles, so we get a simple factor of $4\pi$. There is, however, the complication that we need to figure out what the momentum of the final state particles is. Because we are treating the bottom as massless, this is the same as the energy of the bottom. Since the top quark is at rest, $p_1^\mu = \left(m_t, 0, 0, 0\right)$, so that $p_1\cdot p_2 = m_t E_b$. From our previous arguments, we therefore have

$$\begin{aligned}2m_t E_b &= m_t^2 - m_h^2\,,\\ E_b &= \frac{m_t^2 - m_h^2}{2m_t}\,.\end{aligned}$$

As mentioned, this is the same as the momentum, so putting it all together, we have

$$\Gamma = \frac{p}{8\pi m_t^2}\left(\alpha^2 + \beta^2\right)\left(m_t^2 - m_h^2\right) = \frac{\alpha^2 + \beta^2}{16\pi m_t^3}\left(m_t^2 - m_h^2\right)^2.$$