Physics 745 - Group Theory

Solution Set 20

1. [10] Prove, using only the commutation relations, the first three identities (2.9) from the notes, namely

$$\left[\mathbf{T}^2, T_a\right] = 0, \quad \left[T_3, T_\pm\right] = \pm T_\pm, \quad \text{and} \quad \mathbf{T}^2 = T_\mp T_\pm + T_3^2 \pm T_3$$

These are, in fact, three, two, and two identities respectively. On the first one, it is very helpful to use the identity $\left[A^2, B\right] = A[A,B] + [A,B]A$**.**

It is simplest to write out the commutation relations we need to prove, and then work them out. The first one is the hardest.

$$\begin{aligned}
\left[\mathbf{T}^2, T_1\right] &= \left[T_1^2 + T_2^2 + T_3^2, T_1\right] \\
&= T_1\left[T_1, T_1\right] + \left[T_1, T_1\right]T_1 + T_2\left[T_2, T_1\right] + \left[T_2, T_1\right]T_2 + T_3\left[T_3, T_1\right] + \left[T_3, T_1\right]T_3 \\
&= 0 + 0 - iT_2T_3 - iT_3T_2 + iT_3T_2 + iT_2T_3 = 0, \\
\left[\mathbf{T}^2, T_2\right] &= \left[T_1^2 + T_2^2 + T_3^2, T_2\right] \\
&= T_1\left[T_1, T_2\right] + \left[T_1, T_2\right]T_1 + T_2\left[T_2, T_2\right] + \left[T_2, T_2\right]T_2 + T_3\left[T_3, T_2\right] + \left[T_3, T_2\right]T_3 \\
&= iT_1T_3 + iT_3T_1 + 0 + 0 - iT_3T_1 - iT_1T_3 = 0, \\
\left[\mathbf{T}^2, T_3\right] &= \left[T_1^2 + T_2^2 + T_3^2, T_3\right] \\
&= T_1\left[T_1, T_3\right] + \left[T_1, T_3\right]T_1 + T_2\left[T_2, T_3\right] + \left[T_2, T_3\right]T_2 + T_3\left[T_3, T_3\right] + \left[T_3, T_3\right]T_3 \\
&= -iT_1T_2 - iT_2T_1 + iT_2T_1 + iT_1T_2 + 0 + 0 = 0.
\end{aligned}$$

The next two aren't nearly as bad.

$$\begin{aligned}
\left[T_3, T_+\right] &= \left[T_3, T_1 + iT_2\right] = iT_2 - i^2T_1 = T_+, \\
\left[T_3, T_-\right] &= \left[T_3, T_1 - iT_2\right] = iT_2 + i^2T_1 = -T_-.
\end{aligned}$$

The last one isn't too difficult either.

$$\begin{aligned}
T_\mp T_\pm + T_3^2 \pm T_3 &= \left(T_1 \mp iT_2\right)\left(T_1 \pm iT_2\right) + T_3^2 \pm T_3 = T_1^2 \mp iT_2T_1 \pm iT_1T_2 + T_2^2 + T_3^2 \pm T_3 \\
&= T_1^2 \pm i\left[T_1, T_2\right] + T_2^2 + T_3^2 \pm T_3 = \mathbf{T}^2 \pm i^2T_3 \pm T_3 = \mathbf{T}^2
\end{aligned}$$

Done!