Physics 745 - Group Theory

Solution Set 18

1. [10] The group $\mathcal{G}$ is one-dimensional, so the elements are described by a single real number *x* that runs from $-\infty$ to ∞, and has coordinate multiplication rule

$$\mu(x,y)=\frac{x+y}{1-xy}$$

(a) [1] What is *e*, such that $\mu(x,e)=\mu(e,x)=x$?

By inspection, we see that if $e = 0$ then $\mu(x,e)=\mu(e,x)=x$.

(b) [3] Check the associative rule explicitly.

We must check

$$\mu(\mu(x,y),z)=\mu(x,\mu(y,z))?$$

$$\frac{\frac{x+y}{1-xy}+z}{1-\frac{x+y}{1-xy}z}=\frac{x+\frac{y+z}{1-yz}}{1-x\frac{y+z}{1-yz}}?$$

$$\frac{x+y+z-zxy}{1-xy-xz-yz}=\frac{x-xyz+y+z}{1-yz-xy-xz}?$$

The final expressions are pretty clearly the same.

(c) [6] Work out the left or right measure for this group (they will be the same), and find the volume of this group. Is this group compact?

The definition of the (left) measure is

$$\int d_LRf(R)=\int_{-\infty}^{\infty}dx\, f(R(x))\left|\frac{\partial\mu(x,y)}{\partial y}\right|_{y=e=0}^{-1}=\int_{-\infty}^{\infty}dx\, f(R(x))\left|\frac{(1-xy)+(x+y)x}{1-xy}\right|_{y=0}^{-1}$$

$$=\int_{-\infty}^{\infty}dx\, f(R(x))\left(1+x^2\right)^{-1}=\int_{-\infty}^{\infty}\frac{dx\, f(R(x))}{1+x^2}$$

where we arbitrarily chose our constant to be $C = 1$. The volume of the group, then, is

$$V=\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\arctan x\Big|_{-\infty}^{\infty}=\pi$$

Of course, the volume you get will depend on your choice of constant *C*. Since it came out finite, the group is compact.