Physics 780 – General Relativity

Solution Set V

52. In class we demonstrated that a general gravity wave can be written as $h_{\mu\nu}(x) = h_{\mu\nu} e^{ik\cdot x} + h^*_{\mu\nu} e^{-ik\cdot x}$, where $h_{\mu\nu}$ is a complex tensor which is assumed to be spacelike only $h_{0\nu} = 0$, and transverse so $k_\mu h^{\mu\nu} = 0$. Let's assume it is traveling in the +z direction, so $k^\mu = (k, 0, 0, k)$.

(a) Substitute this expression into the formula for the gravitational stress-energy in this case, $8\pi G t_{\mu\nu} = -\frac{1}{2} h^{\alpha\sigma} \partial_\mu \partial_\nu h_{\alpha\sigma} - \frac{1}{4} \partial_\mu h^{\alpha\sigma} \partial_\nu h_{\alpha\sigma}$.

We simply substitute it in and start simplifying, so we have

$$\begin{aligned} 8\pi G t_{\mu\nu} &= -\tfrac{1}{2}\left(h^{\alpha\sigma} e^{ik\cdot x} + h^{*\alpha\sigma} e^{-ik\cdot x}\right)\partial_\mu \partial_\nu \left(h_{\alpha\sigma} e^{ik\cdot x} + h^*_{\alpha\sigma} e^{-ik\cdot x}\right) \\ &\quad -\tfrac{1}{4}\partial_\mu \left(h^{\alpha\sigma} e^{ik\cdot x} + h^{*\alpha\sigma} e^{-ik\cdot x}\right)\partial_\nu \left(h_{\alpha\sigma} e^{ik\cdot x} + h^*_{\alpha\sigma} e^{-ik\cdot x}\right) \\ &= \tfrac{1}{2} k_\mu k_\nu \left(h^{\alpha\sigma} e^{ik\cdot x} + h^{*\alpha\sigma} e^{-ik\cdot x}\right)\left(h_{\alpha\sigma} e^{ik\cdot x} + h^*_{\alpha\sigma} e^{-ik\cdot x}\right) \\ &\quad +\tfrac{1}{4} k_\mu k_\nu \left(h^{\alpha\sigma} e^{ik\cdot x} - h^{*\alpha\sigma} e^{-ik\cdot x}\right)\left(h_{\alpha\sigma} e^{ik\cdot x} - h^*_{\alpha\sigma} e^{-ik\cdot x}\right) \\ &= \tfrac{1}{4} k_\mu k_\nu \left(3h^{\alpha\sigma} h_{\alpha\sigma} e^{2ik\cdot x} + 3h^{*\alpha\sigma} h^*_{\alpha\sigma} e^{-2ik\cdot x} + h^{*\alpha\sigma} h_{\alpha\sigma} + h^{\alpha\sigma} h^*_{\alpha\sigma}\right). \end{aligned}$$

(b) Some of the terms now have no space dependance, and others go like $e^{\pm 2ik\cdot x}$. Argue that the terms like $e^{\pm 2ik\cdot x}$ will average to zero if you time average (what is the average value of cos(2*ωt*) and sin(2*ωt*)?).

These terms have expressions like cos(2*ωt*) and sin(2*ωt*), which have an average value of zero, and hence are irrelevant. Hence the only relevant terms are the remaining ones, which looks like $8\pi G \langle t_{\mu\nu} \rangle = \frac{1}{2} k_\mu k_\nu h^*_{\alpha\sigma} h^{\alpha\sigma}$. Nice and simple!

(c) Write an explicit expression for the time-averaged value of $\langle t^{30} \rangle$ in terms of $h_{\mu\nu}$ and $h^*_{\mu\nu}$.

Since $k^3 = \omega$, we have $8\pi G \langle t^{30} \rangle = \frac{1}{2} k^3 k^0 h^*_{\alpha\sigma} h^{\alpha\sigma} = \frac{1}{2} \omega^2 h^*_{\alpha\sigma} h^{\alpha\sigma}$.

(d) If we write $h_{\mu\nu} = h_+ e^+_{\mu\nu} + h_\times e^\times_{\mu\nu}$, write $\langle t^{30} \rangle$ explicitly in terms of h^+ and $h^\times$.

The non-zero components of the two polarization tensors are $e^+_{11} = -e^+_{22} = 1$ and $e^\times_{11} = e^\times_{21} = 1$. It is easy to see that these are real, and that $e^+_{ij} e^+_{ij} = e^\times_{ij} e^\times_{ij} = 2$, while $e^+_{ij} e^\times_{ij} = 0$. We therefore have

$$8\pi G\left\langle t^{30}\right\rangle = \tfrac{1}{2}\omega^2 h^*_{\alpha\sigma}h^{\alpha\sigma} = \tfrac{1}{2}\omega^2\left(h^*_+ e^+_{ij} + h^*_\times e^\times_{ij}\right)\left(h_+ e^+_{ij} + h_\times e^\times_{ij}\right)$$
$$= \tfrac{1}{2}\omega^2\left(h^*_+ h_+ e^+_{ij}e^+_{ij} + h^*_+ h_\times e^+_{ij}e^\times_{ij} + h^*_\times h_+ e^\times_{ij}e^+_{ij} + h^*_\times h_\times e^\times_{ij}e^\times_{ij}\right) = \tfrac{1}{2}\omega^2\left(2h^*_+ h_+ + 0 + 0 + 2h^*_\times h_\times\right),$$
$$\left\langle t^{30}\right\rangle = \frac{\omega^2}{8\pi G}\left(\left|h_+\right|^2 + \left|h_\times\right|^2\right).$$

53. In class we found the following expressions for the magnitude of the gravitational waves in terms of quadrupole moments:

$$h^{00} = k_i k_j Q^{ij} + \omega^2 Q^{ii}, \quad h^{0i} = h^{i0} = 2Q^{ij}\omega k_j, \quad h^{ij} = 2\omega^2 Q^{ij} + \delta^{ij}\left(k_\ell k_m Q^{\ell m} - \omega^2 Q^{\ell\ell}\right)$$

The four-vector *k* is given by $k^\mu = (\omega, \mathbf{k})$, with $\omega = |\mathbf{k}|$.

(a) As a warm-up, find the trace $h^\mu{}_\mu = \eta_{\mu\nu}h^{\mu\nu}$.

We have

$$h^\mu{}_\mu = \eta_{\mu\nu}h^{\mu\nu} = -h^{00} + h^{ii} = -k_i k_j Q^{ij} - \omega^2 Q^{ii} + 2\omega^2 Q^{ii} + 3\left(k_\ell k_m Q^{\ell m} - \omega^2 Q^{\ell\ell}\right) = 2k_i k_j Q^{ij} - 2\omega^2 Q^{ii}.$$

(b) We now want to start checking the harmonic condition $k_\mu h^{\mu\nu} = \tfrac{1}{2}k^\nu h^\mu{}_\mu$. Check this for the time component $\nu = 0$.

We will work out both sides until the expression is manifestly true:

$$k_\mu h^{\mu 0} = \tfrac{1}{2}k^0 h^\mu{}_\mu,$$
$$k_0 h^{00} + k_i h^{i0} = \tfrac{1}{2}k^0\left(2k_i k_j Q^{ij} - 2\omega^2 Q^{ii}\right),$$
$$-\omega k_i k_j Q^{ij} - \omega^3 Q^{ii} + 2\omega k_i Q^{ij} k_j = \omega k_i k_j Q^{ij} - \omega^3 Q^{ii}.$$

At this point it is pretty easy to see it is true.

(c) Now check it for the space components, $\nu = j$.

We substitute in again, as before

$$k_\mu h^{\mu j} = \tfrac{1}{2}k^j h^\mu{}_\mu,$$
$$k_0 h^{0j} + k_i h^{ij} = \tfrac{1}{2}k^j\left(2k_i k_\ell Q^{i\ell} - 2\omega^2 Q^{ii}\right),$$
$$-2\omega^2 k_i Q^{ij} + k_i\left(2\omega^2 Q^{ij}\right) + k_j\left(k_\ell k_m Q^{\ell m} - \omega^2 Q^{\ell\ell}\right) = k_j\left(k_i k_\ell Q^{i\ell} - \omega^2 Q^{ii}\right)$$

The first two terms cancel, and other than some slight relabeling, the remaining terms are identical on the two sides of the equation, so we are done.