Physics 780 – General Relativity

# Solution Set I

**21. Consider a geodesic in flat 2D space, but working in polar coordinates $ds^2 = d\rho^2 + \rho^2 d\phi^2$. The non-vanishing Christoffel symbols are $\Gamma^\phi_{\rho\phi} = \Gamma^\phi_{\phi\rho} = \rho^{-1}$, $\Gamma^\rho_{\phi\phi} = -\rho$. Because we have space and no time, geodesics are parameterized by *s*, not $\tau$.**

**(a) Write both components of the geodesic equations for $dU^\mu/ds$.**

The geodesic equation is $\frac{d}{ds}U^\mu + \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta = 0$. Written out explicitly, this is

$$0 = \frac{d}{ds}U^\rho + \Gamma^\rho_{\phi\phi}U^\phi U^\phi = \frac{d}{ds}U^\rho - \rho\left(U^\phi\right)^2,$$
$$0 = \frac{d}{ds}U^\phi + \Gamma^\phi_{\phi\rho}U^\phi U^\rho + \Gamma^\phi_{\rho\phi}U^\rho U^\phi = \frac{d}{ds}U^\phi + \frac{2}{\rho}U^\phi U^\rho.$$

**(b) Show that on a geodesic, $\rho^2 U^\phi$ is constant, that is, $\frac{d}{ds}\left(\rho^2 U^\phi\right) = 0$. Hint: on the $dU^\phi/ds$ equation, replace $U^\rho = d\rho/ds$ and then multiply it by $\rho^2$.**

We take the hint, and find

$$0 = \frac{d}{ds}U^\phi + \frac{2}{\rho}U^\phi \frac{d\rho}{ds},$$
$$0 = \rho^2 \frac{d}{ds}U^\phi + 2\rho\frac{d\rho}{ds}U^\phi = \frac{d}{ds}\left(\rho^2 U^\phi\right).$$

**22. Write out $\left[\nabla_\mu, \nabla_\nu\right] g_{\alpha\beta}$ in terms of the Riemann tensor, and then use the fact that the metric has vanishing covariant derivative to show that $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$**

This commutator must vanish, since $\nabla_\mu g_{\alpha\beta} = \nabla_\nu g_{\alpha\beta} = 0$, but we can also evaluate it using the Riemann tensor, so we have

$$0 = \left[\nabla_\mu, \nabla_\nu\right] g_{\alpha\beta} = -R^\lambda{}_{\alpha\mu\nu} g_{\lambda\beta} - R^\lambda{}_{\beta\mu\nu} g_{\alpha\lambda} = -R_{\beta\alpha\mu\nu} - R_{\alpha\beta\mu\nu},$$
$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}.$$

**23. Show the Jacobi identity** $\left[\left[\nabla_\mu,\nabla_\nu\right],\nabla_\alpha\right]+\left[\left[\nabla_\nu,\nabla_\alpha\right],\nabla_\mu\right]+\left[\left[\nabla_\alpha,\nabla_\mu\right],\nabla_\nu\right]=0$

You simply write this all out, and we have

$$
\begin{aligned}
&\left[\left[\nabla_\mu,\nabla_\nu\right],\nabla_\alpha\right]+\left[\left[\nabla_\nu,\nabla_\alpha\right],\nabla_\mu\right]+\left[\left[\nabla_\alpha,\nabla_\mu\right],\nabla_\nu\right]\\
&=\left[\nabla_\mu,\nabla_\nu\right]\nabla_\alpha-\nabla_\alpha\left[\nabla_\mu,\nabla_\nu\right]+\left[\nabla_\nu,\nabla_\alpha\right]\nabla_\mu-\nabla_\mu\left[\nabla_\nu,\nabla_\alpha\right]+\left[\nabla_\alpha,\nabla_\mu\right]\nabla_\nu-\nabla_\nu\left[\nabla_\alpha,\nabla_\mu\right]\\
&=\nabla_\mu\nabla_\nu\nabla_\alpha-\nabla_\nu\nabla_\mu\nabla_\alpha-\nabla_\alpha\nabla_\mu\nabla_\nu+\nabla_\alpha\nabla_\nu\nabla_\mu+\nabla_\nu\nabla_\alpha\nabla_\mu-\nabla_\alpha\nabla_\nu\nabla_\mu\\
&\quad-\nabla_\mu\nabla_\nu\nabla_\alpha+\nabla_\mu\nabla_\alpha\nabla_\nu+\nabla_\alpha\nabla_\mu\nabla_\nu-\nabla_\mu\nabla_\alpha\nabla_\nu-\nabla_\nu\nabla_\alpha\nabla_\mu+\nabla_\nu\nabla_\mu\nabla_\alpha\\
&=0
\end{aligned}
$$

All the terms cancel.

**24. By letting the Jacobi identity act on a scalar $\phi$, show that $R^\beta{}_{\alpha\mu\nu}+R^\beta{}_{\mu\nu\alpha}+R^\beta{}_{\nu\alpha\mu}=0$.**

Focusing first on just the first term, we have

$$\left[\left[\nabla_\mu,\nabla_\nu\right],\nabla_\alpha\right]\phi=\left[\nabla_\mu,\nabla_\nu\right]\nabla_\alpha\phi-\nabla_\alpha\left[\nabla_\mu,\nabla_\nu\right]\phi=R^\beta{}_{\alpha\mu\nu}\nabla_\beta\phi.$$

If we add these three terms cyclically, we must get zero, so we have

$$0=\left[\left[\nabla_\mu,\nabla_\nu\right],\nabla_\alpha\right]\phi+\left[\left[\nabla_\nu,\nabla_\alpha\right],\nabla_\mu\right]\phi+\left[\left[\nabla_\alpha,\nabla_\mu\right],\nabla_\nu\right]\phi=\left(R^\beta{}_{\alpha\mu\nu}+R^\beta{}_{\mu\nu\alpha}+R^\beta{}_{\nu\alpha\mu}\right)\nabla_\beta\phi$$

The only way this can be true for any scalar is if the object in parentheses vanishes, so

$$R^\beta{}_{\alpha\mu\nu}+R^\beta{}_{\mu\nu\alpha}+R^\beta{}_{\nu\alpha\mu}=0.$$