Physics 780 – General Relativity

Solutions to Homework C

7. [10] Each of the following formulas is true for an appropriate value of *k* in flat 4D-spacetime. In each case, find *k*:

(a) $\eta_{\mu\nu}\eta^{\mu\nu} = k$

(b) $\eta_{\mu\nu}\eta_{\alpha\beta}\eta^{\mu\gamma}\eta^{\beta\alpha}\delta^{\nu}_{\gamma} = k$

(c) $\tilde{\varepsilon}_{\mu\nu\alpha\beta} = k\tilde{\varepsilon}^{\mu\nu\alpha\beta}$

(d) $\tilde{\varepsilon}_{\mu\nu\alpha\beta}\tilde{\varepsilon}^{\mu\nu\alpha\beta} = k$

(e) $\tilde{\varepsilon}_{\mu\nu\alpha\beta}\eta^{\mu\nu} = k\eta_{\alpha\beta}$

Keeping in mind that repeated indices are summed on, a lot of these aren't hard. For the first two, we have $\eta_{\mu\nu}\eta^{\mu\nu} = \delta^{\mu}_{\mu} = 4 = k$ and $\eta_{\mu\nu}\eta_{\alpha\beta}\eta^{\mu\gamma}\eta^{\beta\alpha}\delta^{\nu}_{\gamma} = \delta^{\gamma}_{\nu}\delta^{\alpha}_{\alpha}\delta^{\nu}_{\gamma} = 4\delta^{\nu}_{\nu} = 16 = k$. For the third one, we argued that each side will vanish unless all the indices are unique, so that $\mu\nu\alpha\beta = 0123$ in some order. For the non-vanishing ones, keeping in mind that the metric is diagonal, we would have

$$\tilde{\varepsilon}_{\mu\nu\alpha\beta} = \eta_{\mu\mu'}\eta_{\nu\nu'}\eta_{\alpha\alpha'}\eta_{\beta\beta'}\tilde{\varepsilon}^{\mu'\nu'\alpha'\beta'} = \eta_{\mu\mu}\eta_{\nu\nu}\eta_{\alpha\alpha}\eta_{\beta\beta}\tilde{\varepsilon}^{\mu\nu\alpha\beta} \text{ (no sum) } = \eta_{00}\eta_{11}\eta_{22}\eta_{33}\tilde{\varepsilon}^{\mu\nu\alpha\beta} = -\tilde{\varepsilon}^{\mu\nu\alpha\beta},$$

so $k = -1$. For (d), we note that both factors are zero if any of the indices match each other, so when doing the sum, we only need to keep the terms where $\mu\nu\alpha\beta$ is a permutation of 0123. When this is true, one of the two factors is ± 1 and the other is the negative of this, ∓ 1, so the product is –1. Hence we get one term of –1 for each permutation of 0213, and there are 4! = 24 of them, so the sum is $k = -24$.

For the final one, note that if $\mu \neq \nu$, then $\eta_{\mu\nu} = 0$, and if $\mu = \nu$ then $\tilde{\varepsilon}_{\mu\nu\alpha\beta} = 0$, so every term in the sum is zero, so that $k = 0$ works.

8. **[10] This problem has to do with Maxwell's equations**
   (a) Show that Maxwell's first equation, $\partial_\nu F^{\mu\nu} = J^\mu / \varepsilon_0$, automatically assures that current is conserved, $\partial_\mu J^\mu = 0$.

We will use the fact that F is antisymmetric, that partial derivatives commute, and we can always rename our indices to be whatever we want, so we will swap $\mu \leftrightarrow \nu$ at the second step. We have

$$\partial_\mu J^\mu = \varepsilon_0 \partial_\mu \partial_\nu F^{\mu\nu} = \varepsilon_0 \partial_\nu \partial_\mu F^{\nu\mu} = \varepsilon_0 \partial_\mu \partial_\nu F^{\nu\mu} = -\varepsilon_0 \partial_\mu \partial_\nu F^{\mu\nu} .$$

We note that this expression is equal to minus itself, which is only possible if it vanishes, so $\partial_\mu J^\mu = 0$.

   (b) It is common to write the electromagnetic field tensor in the form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is the four-vector potential. Show that if you do this then the second Maxwell equation is automatically satisfied.

One way to write this expression is as below, and we simply use the commuting property of partial derivatives to show that

$$\begin{aligned}\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} &= \partial_\alpha \partial_\mu A_\nu - \partial_\alpha \partial_\nu A_\mu + \partial_\mu \partial_\nu A_\alpha - \partial_\mu \partial_\alpha A_\nu + \partial_\nu \partial_\alpha A_\mu - \partial_\nu \partial_\mu A_\alpha \\ &= \left(\partial_\alpha \partial_\mu - \partial_\mu \partial_\alpha\right) A_\nu + \left(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu\right) A_\alpha + \left(\partial_\nu \partial_\alpha - \partial_\alpha \partial_\nu\right) A_\mu = 0.\end{aligned}$$

9. **[15] A particle of charge *q* and mass *m* is initially moving with velocity $\mathbf{v} = (v_1, 0, v_2)$. It is placed in a region with a uniform magnetic field in the *z*-direction $B_3 = B$.**
   (a) What is the initial four-velocity $U^\mu(\tau = 0)$?

This is straightforward, $U^\mu = (\gamma, \gamma v_1, 0, \gamma v_2)$, where $\gamma = \left(1 - v_1^2 - v_2^2\right)^{-1/2}$.

   (b) Write down differential equations for all four components of the four velocity $dU^\mu / d\tau$. Solve these equations, subject to the initial conditions, for U^0 and U^3.

The only non-zero part of the electromagnetic tensor are $F_{12} = -F_{21} = B$. We will need $F^\mu{}_\nu$, which introduces a minus sign if the first index is zero, but these components are already zero, so $F^1{}_2 = -F^2{}_1 = B$. We then write out explicitly the equations for $m\, dU^\mu / d\tau = qF^\mu{}_\nu U^\nu$, which we write as a matrix equation:

$$m \frac{d}{d\tau} \begin{pmatrix} U^0 \\ U^1 \\ U^2 \\ U^3 \end{pmatrix} = q \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} U^0 \\ U^1 \\ U^2 \\ U^3 \end{pmatrix}.$$

We now simply write all these out explicitly, dividing by m on both sides to obtain

$$\frac{d}{d\tau}U^0 = 0, \quad \frac{d}{d\tau}U^1 = \frac{qB}{m}U^2, \quad \frac{d}{d\tau}U^2 = -\frac{qB}{m}U^1, \quad \frac{d}{d\tau}U^3 = 0.$$

It is obvious that U^0 and U^3 are constant, and therefore we have

$$U^0 = \gamma, \quad U^3 = \gamma v_2.$$

(c) Find a second order differential equation for U^2 of the form $d^2\left(U^2\right)/d\tau^2 = -\omega^2 U^2$. What is ω?

Taking the derivative of the first derivative, we have

$$\frac{d^2}{d\tau^2}U^2 = \frac{qB}{m}\frac{d}{d\tau}U^1 = -\left(\frac{qB}{m}\right)^2 U^2.$$

(d) Solve the equation for part (c), subject to the initial conditions. There should be one unknown parameter describing U^2 at this point.

This formula says that the second derivative of U^2 is proportional to the negative of the function itself. It is easy to see that the solutions to these equations are

$$U^2 = A\cos\left(\omega\tau\right) + B\sin\left(\omega\tau\right), \quad \text{where} \quad \omega = \frac{qB}{m}.$$

However, the value of U^2 at the start is 0, so we conclude that the A term is unacceptable, so $U^2 = B\sin\left(\omega\tau\right)$.

(e) Using the formula for $dU^2/d\tau$, find a formula for U^1. By matching the initial conditions, you should now have all components of U^μ as a function of τ.

We have $dU^2/d\tau = -\omega U^1$, which tells us that $B\omega\cos\left(\omega\tau\right) = -\omega U^1$, so $U^1 = -B\omega\cos\left(\omega\tau\right)$. But matching the value at $\tau = 0$ then tells us $B = -\gamma v_1$. In summary, we have

$$U^0 = \gamma, \quad U^1 = \gamma v_1\cos\left(\omega\tau\right), \quad U^2 = -\gamma v_1\sin\left(\omega\tau\right), \quad U^3 = \gamma v_2, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - v_1^2 - v_3^2}}.$$