Physics 780 – General Relativity

Solutions to Homework A

1. [5] Using the convention that $c = 1$, rewrite the following quantities (on personal ones, you can lie, but your lie has to be plausible)
(a) [1.5] Your age in m
(b) [1.5] The length of your foot in ns
(c) [2] The Schwarzschild radius of the Sun, given by $2GM$, in km

Your answer will depend on your physical characteristic and what star your home planet orbits, but for Dr. Carlson he is about 60.8 year old, his foot is about a foot long, and the Sun's mass is about 1.989×10^{30} kg , so

$$t = tc = (60.8 \text{ y})(3.156\times10^{7} \text{ s/y})(2.998\times10^{8} \text{ m/s}) = 5.75\times10^{17} \text{ m},$$

$$L = \frac{L}{c} = \frac{(12.0 \text{ in})(0.0254 \text{ m/in})}{2.998\times10^{8} \text{ m/s}} = 1.017\times10^{-9} \text{ s} = 1.017 \text{ ns},$$

$$2GM = \frac{2GM}{c^2} = \frac{2(6.674\times10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-3})(1.989\times10^{30} \text{ kg})}{(2.998\times10^{8} \text{ m/s})^2} = 2954 \text{ m} = 2.954 \text{ km}.$$

2. [5] A particle moves in a helical path given by $(x, y, z) = (R\cos(\omega t), R\sin(\omega t), vt)$. Find a relation between the coordinate time t and the proper time τ, and then rewrite all four coordinates in terms of τ.

We have

$$\begin{aligned} d\tau^2 &= dt^2 - dx^2 - dy^2 - dz^2 = dt^2 - (-R\omega\sin(\omega t)\,dt)^2 - (R\omega\cos(\omega t)\,dt)^2 - (v\,dt)^2 \\ &= dt^2\left[1 - R^2\omega^2\left(\sin^2(\omega t) + \cos^2(\omega t)\right) - v^2\right] = dt^2\left(1 - R^2\omega^2 - v^2\right), \\ \tau &= \int dt\sqrt{1 - R^2\omega^2 - v^2} = t\sqrt{1 - R^2\omega^2 - v^2}. \end{aligned}$$

We can rewrite this as $t = \gamma\tau$, where $\gamma = \dfrac{1}{\sqrt{1 - R^2\omega^2 - v^2}}$, then it is clear that our coordinates are

$$t = \gamma\tau, \quad x = R\cos(\omega\gamma\tau), \quad y = R\sin(\omega\gamma\tau), \quad z = \gamma v\tau, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \omega^2R^2 - v^2}}.$$

3. **[10] A general Lorentz transformation is a 4×4 matrix satisfying** $\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}$**, which can be written in matrix form as** $\eta = \Lambda^T \eta \Lambda$

   (a) [3] Using the formulas $\det(AB) = \det(A)\det(B)$ **and** $\det(A^T) = \det(A)$**, show that** $\det(\Lambda) = \pm 1$**.**

   Working in the matrix notation, we have

$$\det(\eta) = \det(\Lambda^T)\det(\eta)\det(\Lambda) = \det(\eta)\left[\det(\Lambda)\right]^2,$$
$$\left[\det(\Lambda)\right]^2 = 1, \quad \det(\Lambda) = \pm 1.$$

   (b) [3] Using this equation in the case $\mu = \nu = 0$**, show that** $\Lambda^0{}_0 \geq 1$ **or** $\Lambda^0{}_0 \leq -1$**.**

   Using the fact that $\eta_{\alpha\beta}$ has only diagonal elements, the equation for $\mu = \nu = 0$ yields

$$-1 = \eta_{00} = \eta_{\alpha\beta}\Lambda^{\alpha}{}_0\Lambda^{\beta}{}_0 = \eta_{00}\Lambda^0{}_0\Lambda^0{}_0 + \eta_{11}\Lambda^1{}_0\Lambda^1{}_0 + \eta_{22}\Lambda^2{}_0\Lambda^2{}_0 + \eta_{33}\Lambda^3{}_0\Lambda^3{}_0$$
$$= -\left(\Lambda^0{}_0\right)^2 + \left(\Lambda^1{}_0\right)^2 + \left(\Lambda^2{}_0\right)^2 + \left(\Lambda^3{}_0\right)^2,$$
$$\left(\Lambda^0{}_0\right)^2 = 1 + \left(\Lambda^1{}_0\right)^2 + \left(\Lambda^2{}_0\right)^2 + \left(\Lambda^3{}_0\right)^2$$

It follows that $\left(\Lambda^0{}_0\right)^2 \geq 1$, so $\left|\Lambda^0{}_0\right| \geq 1$ which implies $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$.

   (c) [2] Argue that if you start with the identity Lorentz transformation ($\Lambda = 1$**), and then continuously change it, by making small rotations or boosts, the sign of** $\det(\Lambda)$ **and** $\Lambda^0{}_0$ **will never change. Call these Lorentz transformations *proper* Lorentz transformation.**

As we perform small changes to Λ, the determinant and the value of $\Lambda^0{}_0$ can only change in a continuous manner. Therefore the determinant, constrained to the values ±1, cannot jump from positive to negative, so it must always have the value +1. Similarly, $\Lambda^0{}_0$ starts at +1, and since it can't slowly change to a value slightly less than one, it will always have $\Lambda^0{}_0 \geq 1$, also positive.

   (d) [2] Show that time reversal $\Lambda = \mathcal{T} = \mathrm{diag}(-1,1,1,1)$ **parity,** $\Lambda = \mathcal{P} = \mathrm{diag}(1,-1,-1,-1)$**, and the combination** $\Lambda = \mathcal{PT}$ **are improper Lorentz transformations.**

It is trivial to see that $\mathcal{T}$ and $\mathcal{P}$ both have determinant minus one, and therefore are improper. The combination $\mathcal{PT}$ has determinant +1, but it has $\Lambda^0{}_0 = -1$ (as also does $\mathcal{T}$), and hence it is also improper. Hence you can't by small changes turn around in time, turn around like a mirror image, nor can you do both.

4. **[5] The concept of future and past do not work exactly the same in special relativity, but some things are the same. We will say that a point x^μ is in the absolute future of another point y^μ if $x^0 - y^0 > |\mathbf{x} - \mathbf{y}|$. In standard physics, the future of the future is the future. Show that in relativity, if *x* is in the future of *y* and *y* is in the future of *z*, then *x* is in the future of *z*. You may want to look up the triangle inequality.**

We know that $x^0 - y^0 > |\mathbf{x} - \mathbf{y}|$ and $y^0 - z^0 > |\mathbf{y} - \mathbf{z}|$. Adding these two inequalities, we have $x^0 - z^0 > |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$. We want to prove that $x^0 - z^0 > |\mathbf{x} - \mathbf{z}|$. All the we need to finish the proof is to show that

$$|\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}| \ge |\mathbf{x} - \mathbf{z}|.$$

This theorem is the triangle inequality, since it says that the sum of two sides of a triangle is always at least as long as the third side. Mathematically, you can prove this by first defining $\mathbf{a} = \mathbf{x} - \mathbf{y}$ and $\mathbf{b} = \mathbf{y} - \mathbf{z}$. We then substitute this into the expression, and then square it, so we are trying to prove

$$|\mathbf{a}| + |\mathbf{b}| \ge |\mathbf{a} + \mathbf{b}|,$$
$$a^2 + 2ab + b^2 \ge (\mathbf{a} + \mathbf{b})^2,$$
$$a^2 + 2ab + b^2 \ge a^2 + 2\mathbf{a} \cdot \mathbf{b} + b^2,$$
$$ab \ge \mathbf{a} \cdot \mathbf{b} = ab\cos\theta.$$

Since we always have $\cos\theta \le 1$, this is obvious.