Physics 780 – General Relativity

# Homework Set O

36. In homework set L, problem 30, you found the general solution for a black hole if there is also a cosmological constant. In the problem, we are going to consider a universe with no black hole and just a cosmological constant, with metric

$$ds^2 = -\left(1-\tfrac{1}{3}\Lambda r^2\right)\mathrm{d}t^2 + \left(1-\tfrac{1}{3}\Lambda r^2\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right).$$

Our ultimate goal is to change coordinates to get rid of the apparent singularity, and make a Penrose diagram for this metric.

(a) This metric has an apparent singularity at $r = b$ (what is $b$?). Rewrite the metric in terms of $b$ instead of $\Lambda$. In which regions of radius $r \in (0,\infty)$ are $r$ and $t$ spacelike or timelike?

(b) As we did for Schwarzschild, define a coordinate $r^* = r^*(r)$ such that light-like radial curves will have $dr^*/dt = \pm 1$, *i.e.*, at 45° angles. This will require an integration; choose the constant of integration so that $r^* = 0$ when $r = 0$. What value of $r^*$ corresponds to the trouble spot $r = b$?

(c) Unlike Schwarzschild, it is easy to invert this relation, so we can find $r = r(r^*)$. Use this to write the metric entirely in terms of $r^*$. Then change variables to null coordinates $t, r^* \to u, v$, where $v = t + r^*$ and $u = t - r^*$. In $u$, $v$ coordinates, where is $r = 0$ now? In $u$, $v$ coordinates, where is $r = b$ now? Write the metric in terms of $u$ and $v$.

(d) In an attempt to get $r = b$ back under control, define new coordinates $v' = -e^{-v/b}$ and $u' = e^{u/b}$. Write the metric in terms of $u'$ and $v'$. Write a formula for $r$ in terms of $u'$ and $v'$. Write the metric in terms of $u'$ and $v'$. What is the equation for the points that correspond to $r = 0$? To $r = b$? To $r = \infty$?

(e) Define new coordinates $u'' = \tan^{-1}(u')$, $v'' = \tan^{-1}(v')$. Write the metric in terms of $u''$ and $v''$. For this final step, eliminate $b$ and go back to $\Lambda$ for the metric.

(f) Make a final change of coordinates to $u'', v'' \to R, T$, where $v'' = \tfrac{1}{2}(T + R)$ and $u'' = \tfrac{1}{2}(T - R)$. Write the metric in terms of $T$ and $R$. In $(T,R)$ space, where are the locations $r = 0$, $b$, and $\infty$? Make a Penrose diagram in $(T,R)$ coordinates, with these three values of $r$ marked as one or more lines.

Possibly Helpful Formulas: $\displaystyle\int \frac{dx}{b^2 - x^2} = \frac{1}{b}\tanh^{-1}\left(\frac{x}{b}\right)$, $\displaystyle\frac{d}{d\psi}\tanh\psi = \operatorname{sech}^2\psi$

$$\tanh^2\psi + \operatorname{sech}^2\psi = 1, \quad \tanh\psi = \frac{e^{\psi} - e^{-\psi}}{e^{\psi} + e^{-\psi}}, \quad \operatorname{sech}\psi = \frac{2}{e^{\psi} + e^{-\psi}},$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$