Physics 712 – Electricity and Magnetism
Midterm Exam, Spring 2016

Please note that some possibly helpful formulas and integrals appear on the second page. Each question is worth 20 points.

1. A charge distribution in vacuum produces a potential given, in cylindrical coordinates, by

$$\Phi(\rho,\phi,z)=\frac{\alpha}{\varepsilon_0}\ln\left(1+\frac{a^2}{\rho^2}\right);$$ that is, it is independent of both z and ϕ. Find the electric field E

everywhere and the charge density $\rho(\mathbf{x})$ everywhere away from the z-axis. Don't confuse the charge density with the cylindrical coordinate ρ. Demonstrate that there is also a linear charge density λ along the z-axis, and determine its magnitude.

2. The potential on a sphere of radius a is given by $\Phi(a,\theta,\phi)=V\cos\theta$. The potential on a sphere of radius $2a$ is given by $\Phi(2a,\theta,\phi)=V$. Find the potential in the region $a<r<2a$ assuming there are no charges between the two spheres.
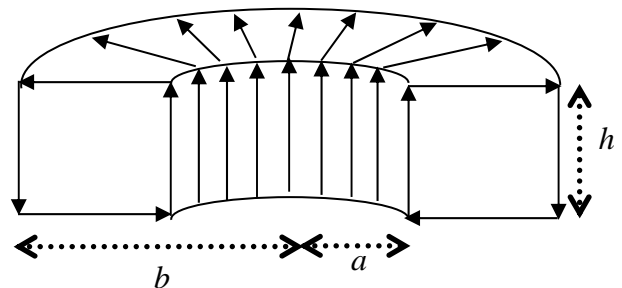
3. Two matching point charges q are placed on opposite sides at a distance r from the center of a conducting sphere of radius a. Find the total force on one of the charges if the sphere is (a) grounded (b) neutral.
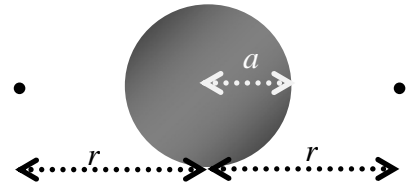
4. A solid conducting sphere of radius a is surrounded by a hollow conducting sphere of radius c. A charge Q is placed on the inner sphere and $–Q$ on the outer sphere. The region from $r = a$ to $r = b$ is filled with an insulator with dielectric constant ε, and the rest is left in vacuum. Find **D** and **E** everywhere between the two spheres, and the potential difference $\Delta\Phi=\Phi_c-\Phi_a$. Also find the bound surface charge density at $r = b$, and the total energy.
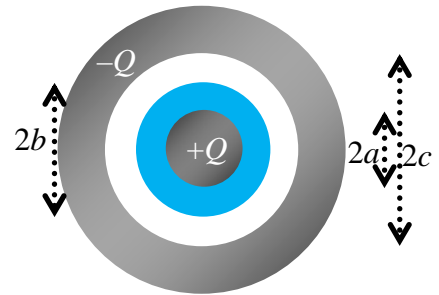
5. A tokamak is in the shape of a rectangular cross-section donut centered on the z-axis, with height h, inner radius a and outer radius b, as sketched in the cutaway view at right. A total current I is then sent around the rectangular direction of the tokamak, so it goes up through the hole in the center, across the top, down on the outside, and then back to the center, equally at all angles, so as to generate a magnetic flux density $\mathbf{B}(\mathbf{x})=B(\rho,z)\hat{\boldsymbol{\phi}}$ in cylindrical coordinates. Find **B** at all points inside or outside the tokamak, and find the total magnetic energy stored inside the tokamak.

Possibly helpful formulas

Multipole Expansion	**General Solution of Laplace in Spherical**
$q_{lm} = \int Y_{lm}^*(\theta,\phi) r^l \rho(\mathbf{x}) d^3\mathbf{x}$ $$\Phi(\mathbf{x}) = \frac{1}{\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{(2l+1)r^{l+1}}\sum_{m=-l}^{l} q_{lm}Y_{lm}(\theta,\phi)$$	$$\Phi(\mathbf{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + B_{lm}r^{-l-1}\right)Y_{lm}(\theta,\phi)$$

Cylindrical Coordinates	**Spherical Harmonics**	**Polarization and Magnetization**
$\int f d^3\mathbf{x} = \int d\phi \int dz \int f\,\rho\, d\rho$ $\nabla f = \hat{\boldsymbol{\rho}}\dfrac{\partial f}{\partial \rho} + \hat{\boldsymbol{\phi}}\dfrac{1}{\rho}\dfrac{\partial f}{\partial \phi} + \hat{\mathbf{z}}\dfrac{\partial f}{\partial z}$ $\nabla \cdot \mathbf{A} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho A_\rho\right) + \dfrac{1}{\rho}\dfrac{\partial A_\phi}{\partial \phi} + \dfrac{\partial A_z}{\partial z}$	$Y_{00} = \dfrac{1}{\sqrt{4\pi}}$ $Y_{10} = \sqrt{\dfrac{3}{4\pi}}\cos\theta$ $Y_{1,\pm 1} = \mp\sqrt{\dfrac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$	$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ $\mathbf{B} = \mu_0\left(\mathbf{H} + \mathbf{M}\right)$ $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$

Method of Images for Spheres

$$q' = -\frac{aq}{r}, \quad r' = \frac{a^2}{r}$$