Physics 712
Chapter X Problems

5. Suppose you are at sufficiently high frequency that you can treat the free electrons in a conductor as a plasma.

(a) Find a formula for the index of refraction if $\omega < \omega_p$. Show that, at all angles and both polarizations, the intensity of the reflected light matches that of the incident light.

When we treat the electrons as a plasma, then the effective permittivity is given simply by $\varepsilon_{\text{eff}} = \varepsilon_0 \left(1 - \omega_p^2 / \omega^2\right)$. The index of refraction is then

$$n = \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \frac{1}{\omega} \sqrt{\omega^2 - \omega_p^2} = \frac{i}{\omega} \sqrt{\omega_p^2 - \omega^2} .$$

Hence it will be pure imaginary for frequencies $\omega < \omega_p$. Suppose you are trying to move into a region of plasma, so that this is the formula for n' from a region with normal n. The amplitude of the reflected wave, compared to the incident wave, for perpendicular, is given by

$$\frac{E''}{E} = \frac{n \cos\theta - \sqrt{n'^2 - n^2 \sin^2\theta}}{n \cos\theta + \sqrt{n'^2 - n^2 \sin^2\theta}} = \frac{n \cos\theta - i\sqrt{n^2 \sin^2\theta - 1 + \omega_p^2/\omega^2}}{n \cos\theta + i\sqrt{n^2 \sin^2\theta - 1 + \omega_p^2/\omega^2}}.$$

The expression under the radical is now plainly positive, so it is obvious the numerator and denominators are complex conjugates of each other. Indeed, if you multiply this by its complex conjugate, the resulting numerator and denominator are immediately identical.

For parallel, the corresponding formula is

$$\frac{E''}{E} = \frac{n'^2 \cos\theta - n\sqrt{n'^2 - n^2 \sin^2\theta}}{n'^2 \cos\theta + n\sqrt{n'^2 - n^2 \sin^2\theta}} = \frac{\left(1 - \omega_p^2/\omega^2\right)\cos\theta - in\sqrt{n^2 \sin^2\theta + \omega_p^2/\omega^2 - 1}}{\left(1 - \omega_p^2/\omega^2\right)\cos\theta + in\sqrt{n^2 \sin^2\theta + \omega_p^2/\omega^2 - 1}}.$$

Again, the numerator and denominator are complex conjugates of each other.

(b) For normal incidence and $\omega < \omega_p$, find a formula for how the intensity drops off in the conductor as a function of distance within the conductor.

The frequency ω remains the same. Inside the conductor, we have

$$k = \frac{n}{c}\omega = \frac{\omega}{c} i \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \frac{i}{c}\sqrt{\omega^2 - \omega_p^2} .$$

The wave number of the evanescent wave along the conducting surface will be identical to the incoming wave, but for normal incidence, this just implies $k_x = 0$, so the k we just computed is k_z. The electric field is proportional to $e^{i\mathbf{k}\cdot\mathbf{x}}$, which in this case implies

$$\mathbf{E}' \sim \mathbf{E}'_0 e^{i\mathbf{k}\cdot\mathbf{x}} = \mathbf{E}'_0 \exp\left[i^2 \frac{z}{c}\sqrt{\omega_p^2 - \omega^2}\right] = \mathbf{E}'_0 \exp\left(-z\sqrt{\omega_p^2 - \omega^2}\Big/c\right)$$

The intensity is proportional to the square of the electric field, so

$$I' = I \exp\left(-2z\sqrt{\omega_p^2 - \omega^2}\Big/c\right)$$

(c) Assume that in aluminum, there is one conduction electron per atom. Find the plasma frequency for aluminum. For visible light with vacuum wavelength of 500 nm, how far into an aluminum mirror must you go before the power drops by a factor of 10^5?

Aluminum has an atomic weight of 26.98 g/mol and a density of 2.70 g/cm^3, so that we have a number density of

$$n_{Al} = \frac{\left(2.70 \text{ g/cm}^3\right)\left(10^6 \text{ m}^3/\text{cm}^3\right)\left(6.022\times10^{23} \text{ atoms/mol}\right)}{\left(26.98 \text{ g/mol}\right)} = 6.027\times10^{28} \text{ atoms/m}^3 .$$

This is, by assumption, the same density as the density of free electrons. We therefore have

$$\omega_p^2 = \frac{n_e e^2}{\varepsilon_0 m_e} = \frac{\left(6.027\times10^{28} \text{ m}^{-3}\right)\left(1.602\times10^{-19}\text{C}\right)^2}{\left(8.854\times10^{-12} \text{ m}^{-3}\text{kg}^{-1}\text{s}^2\text{C}^2\right)\left(9.109\times10^{-31} \text{ kg}\right)} = 1.918\times10^{32} \text{ s}^{-2} ,$$

$$\omega_p = 1.385\times10^{16} \text{ s}^{-1} .$$

The angular frequency for light of 500 nm is

$$\omega = 2\pi f = \frac{2\pi c}{\lambda} = \frac{2\pi\left(2.998\times10^8 \text{ m/s}\right)}{500\times10^{-9} \text{ m}} = 3.77\times10^{15} \text{ s}^{-1} .$$

This is down by a factor of 3.5 from the plasma frequency.
We want the intensity to fall off by a factor of 10^5, so we want

$$10^{-5} = \frac{I'}{I} = \exp\left(-2z\sqrt{\omega_p^2 - \omega^2}\Big/c\right),$$

$$z\frac{2}{c}\sqrt{\omega_p^2 - \omega^2} = \ln 10^5 = 11.51,$$

$$z = \frac{11.51c}{2\sqrt{\omega_p^2 - \omega^2}} = \frac{11.51\left(2.998\times10^8 \text{ m/s}\right)}{2\sqrt{1.918\times10^{32} \text{ s}^{-2} - \left(3.77\times10^{15} \text{ s}^{-1}\right)^2}}$$

$$= \frac{11.51\left(2.998\times10^8 \text{ m/s}\right)}{2\left(1.333\times10^{16} \text{ s}^{-1}\right)} = 1.29\times10^{-7} \text{ m} = 129 \text{ nm} .$$

Remarkably, it need only be about one-fourth of a wavelength thick to completely reflect a visible light wave.