Physics 712
Chapter 8 Solutions

5. [15] For a box-shaped conducting cavity of dimensions $a\times b\times c$, with *c* along the conventional *z*-direction, work out explicitly every non-zero component for the $\mathrm{TM}_{1,1,0}$ mode. Then calculate the total electric and magnetic energy in the cavity as a function of time. As a check, show the total is independent of time.

The TM modes have longitudinal magnetic fields of the form $E_z=\psi(x,y)\cos(kz)e^{-i\omega t}$, where $k=\pi p/d$. But this mode has $p = 0$, so the cosine factor is just 1. The function $\psi(x,y)$ can be found in the TM modes for wave guides, and we find

$$E_z=E_0\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi my}{b}\right)e^{-i\omega t}=E_0\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)e^{-i\omega t}$$

We will also need γ and ω, which are given by

$$\gamma_{11}^2=\frac{\pi^2}{a^2}+\frac{\pi^2}{b^2}=\pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right),\quad \gamma=\frac{\pi ab}{\sqrt{a^2+b^2}},$$
$$\mu\varepsilon\omega_{110}^2=\frac{\pi^2}{a^2}+\frac{\pi^2}{b^2}+0=\gamma_{11}^2,\quad \omega=\frac{1}{\sqrt{\mu\varepsilon}}\gamma.$$

We still need to find the transverse components, which are given by

$$\begin{aligned}
\mathbf{E}_t&=-\gamma^{-2}k\nabla_t\psi(x,y)\sin(kz)e^{-i\omega t}=0,\\
\mathbf{B}_t&=i\omega\varepsilon\mu\gamma^{-2}\hat{\mathbf{z}}\times\nabla_t\psi(x,y)\cos(kz)e^{-i\omega t}\\
&=i\gamma^{-1}\sqrt{\varepsilon\mu}\,\hat{\mathbf{z}}\times\left(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}\right)E_0\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)e^{-i\omega t}\\
&=\frac{i\sqrt{\varepsilon\mu}\,abE_0}{\pi\sqrt{a^2+b^2}}\hat{\mathbf{z}}\times\left[\frac{\pi}{a}\hat{\mathbf{x}}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)+\frac{\pi}{b}\hat{\mathbf{y}}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)\right]e^{-i\omega t}\\
&=\frac{iE_0\sqrt{\mu\varepsilon}}{\sqrt{a^2+b^2}}\left[b\hat{\mathbf{y}}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)-a\hat{\mathbf{x}}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)\right]e^{-i\omega t}.
\end{aligned}$$

We then need to take the real part. It is easy to see that $\mathrm{Re}\left(e^{-i\omega t}\right)=\cos(\omega t)$ and $\mathrm{Re}\left(ie^{-i\omega t}\right)=\sin(\omega t)$. Summarizing, all our fields are

$$\mathbf{E}=E_0\hat{\mathbf{z}}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\cos(\omega t),$$
$$\mathbf{B}=\frac{E_0\sqrt{\mu\varepsilon}}{\sqrt{a^2+b^2}}\left[b\hat{\mathbf{y}}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)-a\hat{\mathbf{x}}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right)\right]\sin(\omega t)$$

We now work out the electric and magnetic energy as follows:

$$U_E = \tfrac{1}{2}\int_V \mathbf{E}\cdot\mathbf{D}d^3\mathbf{x} = \tfrac{1}{2}\varepsilon\int_V \mathbf{E}^2 d^3\mathbf{x} = \tfrac{1}{2}\varepsilon E_0^2 \cos^2(\omega t)\int_0^a \sin^2\left(\frac{\pi x}{a}\right)dx\int_0^b \sin\left(\frac{\pi y}{b}\right)dy\int_0^d dz$$

$$= \tfrac{1}{2}\varepsilon E_0^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)(d)\cos^2(\omega t) = \tfrac{1}{8}\varepsilon E_0^2 abd\cos^2(\omega t),$$

$$U_B = \tfrac{1}{2}\int_V \mathbf{B}\cdot\mathbf{H}d^3\mathbf{x} = \frac{1}{2\mu}\int_V \mathbf{B}^2 d^3\mathbf{x}$$

$$= \frac{\varepsilon E_0^2 \sin^2(\omega t)}{2\left(a^2+b^2\right)}\left[b^2\int_0^a \cos^2\left(\frac{\pi x}{a}\right)dx\int_0^b \sin^2\left(\frac{\pi y}{b}\right)dy + a^2\int_0^a \sin^2\left(\frac{\pi x}{a}\right)dx\int_0^b \cos^2\left(\frac{\pi y}{b}\right)dy\right]\int_0^d dz$$

$$= \frac{\varepsilon E_0^2}{2\left(a^2+b^2\right)}\sin^2(\omega t)\left[b^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right) + a^2\left(\tfrac{1}{2}a\right)\left(\tfrac{1}{2}b\right)\right]d = \frac{\varepsilon E_0^2 abd\left(a^2+b^2\right)}{8\left(a^2+b^2\right)}\sin^2(\omega t)$$

$$= \tfrac{1}{8}\varepsilon E_0^2 abd\sin^2(\omega t).$$

Obviously, the sum of these two expressions is $\tfrac{1}{8}\varepsilon E_0^2 abd$, independent of time.