Physics 712
Chapter 6 Solutions

2. [5] For question 1, find the total energy flux per unit length flowing out of a cylinder of radius ρ centered on the *z*-axis as a function of time.

The total energy flowing out is just given by

$$\frac{dU}{dt} = \int_S (\mathbf{E}\times\mathbf{H})\cdot\hat{\mathbf{n}}\,da = \frac{1}{\mu_0}\int_0^L dz\int_0^{2\pi}\left[(\mathbf{E}\times\mathbf{B})\cdot\hat{\boldsymbol{\rho}}\right]\rho\,d\phi$$

$$= \frac{L}{\mu_0}\left(-\frac{\mu_0 Ic}{2\pi\sqrt{c^2t^2-\rho^2}}\hat{\mathbf{z}}\right)\times\left(\frac{\mu_0 Ict}{2\pi\rho\sqrt{c^2t^2-\rho^2}}\hat{\boldsymbol{\varphi}}\right)\cdot\hat{\boldsymbol{\rho}}\int_0^{2\pi}\rho\,d\phi = \frac{L\mu_0 I^2c^2t}{2\pi\left(c^2t^2-\rho^2\right)},$$

$$\frac{1}{L}\frac{dU}{dt} = \frac{\mu_0 I^2c^2t}{2\pi\left(c^2t^2-\rho^2\right)}$$

This formula only applies at times $ct > \rho$; at earlier times it is zero. This function diverges when the fields first turn on, which is due to the sudden nature of the current turn on, but it falls to zero at infinite time. It is perhaps not surprising that the time integral diverges, since one can show that the magnetic field in a steady state is infinite.

3. [15] An oscillating point dipole with dipole moment $\mathbf{p} = p\hat{\mathbf{z}}\sin(\omega t)$ at the origin results in scalar and vector potentials (in Lorentz gauge) at large *r* of approximately

$$\Phi = \frac{p\omega\cos\theta}{4\pi\varepsilon_0 rc}\cos(\omega t-\omega r/c), \quad \mathbf{A} = \frac{\mu_0 p\omega}{4\pi r}\cos(\omega t-\omega r/c)\left(\hat{\mathbf{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\right).$$

(a) Find the leading order terms at large *r* for the electric and magnetic fields (these will be terms of order r^{-1}). As a check, E should be entirely in the $\hat{\boldsymbol{\theta}}$ direction and B in the $\hat{\boldsymbol{\varphi}}$ direction.

We start by calculating the electric field in a straightforward manner:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A} = -\hat{\mathbf{r}}\frac{\partial}{\partial r}\Phi - \frac{1}{r}\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta}\Phi - \frac{\partial}{\partial t}\mathbf{A}$$

However, we want to keep only terms that are of order r^{-1}. This means that we should ignore the angle derivatives, and when we take a derivative with respect to *r*, it should act only on the cosine, so as to avoid getting more powers of *r* in the denominator. We therefore have

$$\mathbf{E} = -\hat{\mathbf{r}}\frac{p\omega^2\cos\theta}{4\pi\varepsilon_0 rc^2}\sin\left(\omega t-\frac{\omega r}{c}\right) + \frac{\mu_0 p\omega^2}{4\pi r}\sin\left(\omega t-\frac{\omega r}{c}\right)\left(\hat{\mathbf{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\right)$$

$$= \frac{\mu_0 p\omega^2}{4\pi r}\sin\left(\omega t-\frac{\omega r}{c}\right)\left[-\hat{\mathbf{r}}\cos\theta+\hat{\mathbf{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\right] = -\frac{\mu_0 p\omega^2}{4\pi r}\hat{\boldsymbol{\theta}}\sin\left(\omega t-\frac{\omega r}{c}\right)\sin\theta.$$

We used the relation $\mu_0\varepsilon_0 = c^{-2}$ to simplify this a little. For the magnetic field, we have

$$\mathbf{B} = \nabla\times\mathbf{A} = \hat{\boldsymbol{\varphi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \approx \hat{\boldsymbol{\varphi}}\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) = -\hat{\boldsymbol{\varphi}}\frac{\mu_0 p\omega^2}{4\pi cr}\sin\left(\omega t - \frac{\omega r}{c}\right)\sin\theta .$$

(b) Find the total power flowing out of a sphere of radius r centered on the origin.

The Poynting vector is given by

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0}\left[\frac{\mu_0 p\omega^2}{4\pi r}\hat{\boldsymbol{\theta}}\sin\left(\omega t - \frac{\omega r}{c}\right)\sin\theta\right]\times\left[\hat{\boldsymbol{\varphi}}\frac{\mu_0 p\omega^2}{4\pi cr}\sin\left(\omega t - \frac{\omega r}{c}\right)\sin\theta\right]$$

$$= \frac{\mu_0 p^2\omega^4}{16\pi^2 cr^2}\sin^2\left(\omega t - \frac{\omega r}{c}\right)\sin^2\theta\,\hat{\mathbf{r}}$$

We now simply integrate this over a sphere of radius r to give

$$\frac{dU}{dt} = -\int_S \mathbf{S}\cdot\hat{\mathbf{n}} = -\frac{\mu_0 p^2\omega^4}{16\pi^2 cr^2}\sin^2\left(\omega t - \frac{\omega r}{c}\right)\int \sin^2\theta\, r^2 d\Omega$$

$$= -\frac{\mu_0 p^2\omega^4}{16\pi^2 c}\sin^2\left(\omega t - \frac{\omega r}{c}\right)\int_0^{2\pi} d\phi\int_{-1}^{1}\left(1-\cos^2\theta\right)d\left(\cos\theta\right) = -\frac{\mu_0 p^2\omega^4}{8\pi c}\sin^2\left(\omega t - \frac{\omega r}{c}\right)\frac{4}{3}$$

$$= -\frac{\mu_0 p^2\omega^4}{6\pi c}\sin^2\left(\omega t - \frac{\omega r}{c}\right)$$