Physics 712
Chapter 3 Problems

4. [10] A grounded conducting cube of side *a* has charge density $\rho(\mathbf{x}) = \lambda x(a-x)y(a-y)z(a-z)$ inside it. Find the potential everywhere, and numerically at the center.

We use the Green's function approach, which gives the answer as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V G(\mathbf{x},\mathbf{x}')\,d^3\mathbf{x}'$$
$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\frac{8a^2\lambda}{\pi^2\varepsilon_0 a^3\left(n^2+m^2+p^2\right)}\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi my}{a}\right)\sin\left(\frac{\pi pz}{a}\right)$$
$$\times\int_0^a x'(a-x')\sin\left(\frac{\pi nx'}{a}\right)dx'\int_0^a y'(a-y')\sin\left(\frac{\pi my'}{a}\right)dy'\int_0^a z'(a-z')\sin\left(\frac{\pi pz'}{a}\right)dz'$$

We have three nearly identical integrals to do, which Maple is happy to help us with.

```
> assume(n::integer);int(x*(a-x)*sin(Pi*n*x/a),x=0..a);
```

We find:

$$\int_0^a x'(a-x')\sin\left(\frac{\pi nx'}{a}\right)dx' = \frac{2a^3}{\pi^3 n^3}\left[1-(-1)^n\right] = \begin{cases} 4a^3/\left(\pi^3 n^3\right) & n \text{ odd,} \\ 0 & n \text{ even.}\end{cases}$$

So our potential is

$$\Phi(\mathbf{x}) = \sum_{n\text{ odd}}^{\infty}\sum_{m\text{ odd}}^{\infty}\sum_{p\text{ odd}}^{\infty}\frac{512a^8\lambda}{\pi^{11}\varepsilon_0 n^3 m^3 p^3\left(n^2+m^2+p^2\right)}\sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi my}{a}\right)\sin\left(\frac{\pi pz}{a}\right).$$

We now wish to evaluate this at the center, where each of the sines becomes $\sin\left(\tfrac{1}{2}\pi n\right)$, which alternates between being +1 and –1. We therefore have

$$\Phi\left(\tfrac{1}{2}a,\tfrac{1}{2}a,\tfrac{1}{2}a\right) = \sum_{n\text{ odd}}^{\infty}\sum_{m\text{ odd}}^{\infty}\sum_{p\text{ odd}}^{\infty}\frac{512\lambda a^8(-1)^{(n+m+p-3)/2}}{\pi^{11}\varepsilon_0 n^3 m^3 p^3\left(n^2+m^2+p^2\right)}.$$

Because of the high power in the denominator, this sum converges pretty quickly. We can let Maple do it for us

```
> add(add(add(evalf(512*(-1)^(n+m+p)/(2*n+1)^3/(2*m+1)^3/
  (2*p+1)^3/((2*n+1)^2+(2*m+1)^2+(2*p+1)^2)/Pi^11),n=0..99),
  m=0..99),p=0..99);
```

We find $\Phi\left(\tfrac{1}{2}a,\tfrac{1}{2}a,\tfrac{1}{2}a\right) = 0.0005641353100\,\lambda a^8/\varepsilon_0$. It should be noted that the first term gives a pretty good approximation, which is $\Phi\left(\tfrac{1}{2}a,\tfrac{1}{2}a,\tfrac{1}{2}a\right) \approx 0.000580\,\lambda a^8/\varepsilon_0$, or about 3% off.