Physics 712
Chapter 3 Problems

1. [10] On the sphere *r* = *a*, the potential is given by $\Phi = \lambda x^2$.

(a) Write the potential in terms of linear combinations of spherical harmonics. You will probably find the solution to quantum mechanics problem 7.5 part (b) helpful, which you can find at http://users.wfu.edu/ecarlson/quantum/solutions/sol7_5.pdf

Taking the hint, it was demonstrated there that

$$x^2 = \tfrac{2}{3} r^2 \sqrt{\pi} Y_{00}(\theta,\phi) - \tfrac{2}{3} r^2 \sqrt{\tfrac{\pi}{5}} Y_{20}(\theta,\phi) + \sqrt{\tfrac{2\pi}{15}} r^2 \left[Y_{22}(\theta,\phi) + Y_{2,-2}(\theta,\phi) \right],$$

and therefore, on the surface,

$$\Phi = \tfrac{2}{3} \lambda a^2 \sqrt{\pi} Y_{00}(\theta,\phi) - \tfrac{2}{3} \lambda a^2 \sqrt{\tfrac{\pi}{5}} Y_{20}(\theta,\phi) + \sqrt{\tfrac{2\pi}{15}} \lambda a^2 \left[Y_{22}(\theta,\phi) + Y_{2,-2}(\theta,\phi) \right],$$

(b) Find the potential for *r* > *a* and *r* < *a*. There should only be a finite number of terms in your final answer in each case.

The general solution inside and outside are

$$\Phi_{\text{in}}(\mathbf{x}) = \sum_{l=0}^{\infty} A_{lm} r^l Y_{lm}(\theta,\phi), \quad \Phi_{\text{out}}(\mathbf{x}) = \sum_{l=0}^{\infty} B_{lm} r^{-l-1} Y_{lm}(\theta,\phi)$$

By comparison with our expression on the surface, it is evident that the only terms are when (l,m) takes on one of the four values $(0,0),(2,0),(2,2),(2,-2)$. For the interior solution, we see that

$$A_{00} = \tfrac{2}{3} \lambda a^2 \sqrt{\pi}, \quad A_{20} a^2 = -\tfrac{2}{3} \lambda a^2 \sqrt{\tfrac{\pi}{5}}, \quad A_{2,\pm 2} a^2 = \sqrt{\tfrac{2\pi}{15}} a^2 .$$

For the exterior solution, we have

$$B_{00} a^{-1} = \tfrac{2}{3} \lambda a^2 \sqrt{\pi}, \quad B_{20} a^{-3} = -\tfrac{2}{3} \lambda a^2 \sqrt{\tfrac{\pi}{5}}, \quad B_{2,\pm 2} a^{-3} = \sqrt{\tfrac{2\pi}{15}} a^2 .$$

Substituting these back into our expressions, we find

$$\Phi_{\text{in}}(\mathbf{x}) = \tfrac{2}{3} \lambda a^2 \sqrt{\pi} Y_{00}(\theta,\phi) + \lambda r^2 \left[-\tfrac{2}{3} \sqrt{\tfrac{\pi}{5}} Y_{20}(\theta,\phi) + \sqrt{\tfrac{2\pi}{15}} Y_{22}(\theta,\phi) + \sqrt{\tfrac{2\pi}{15}} Y_{2,-2}(\theta,\phi) \right],$$

$$\Phi_{\text{out}}(\mathbf{x}) = \tfrac{2}{3} \lambda a^3 r^{-1} \sqrt{\pi} Y_{00}(\theta,\phi) + \lambda a^5 r^{-3} \left[-\tfrac{2}{3} \sqrt{\tfrac{\pi}{5}} Y_{20}(\theta,\phi) + \sqrt{\tfrac{2\pi}{15}} Y_{22}(\theta,\phi) + \sqrt{\tfrac{2\pi}{15}} Y_{2,-2}(\theta,\phi) \right].$$

If we want to, we can then substitute the relevant spherical harmonics, which are

$$Y_{00}(\theta,\phi) = \tfrac{1}{\sqrt{4\pi}}, \quad Y_{20}(\theta,\phi) = \sqrt{\tfrac{5}{4\pi}} \left(\tfrac{3}{2} \cos^2\theta - \tfrac{1}{2} \right), \quad Y_{2,\pm 2}(\theta,\phi) = \tfrac{1}{4} \sqrt{\tfrac{15}{2\pi}} \sin^2\theta e^{\pm 2i\phi} .$$

This then yields

$$\Phi_{\text{in}}(\mathbf{x}) = \tfrac{1}{3}\lambda a^2 + \lambda r^2\left[\tfrac{1}{6} - \tfrac{1}{2}\cos^2\theta + \tfrac{1}{2}\sin^2\theta\cos(2\phi)\right],$$
$$\Phi_{\text{out}}(\mathbf{x}) = \tfrac{1}{3}\lambda a^3 r^{-1} + \lambda a^5 r^{-3}\left[\tfrac{1}{6} - \tfrac{1}{2}\cos^2\theta + \tfrac{1}{2}\sin^2\theta\cos(2\phi)\right].$$

If desired, these can be further simplified to

$$\Phi_{\text{in}}(\mathbf{x}) = \tfrac{1}{3}\lambda\left(a^2 - r^2\right) + \lambda r^2\sin^2\theta\cos^2\phi\,, \quad \Phi_{\text{ou}}(\mathbf{x}) = \tfrac{1}{3}\lambda\left(a^3 r^{-1} - a^5 r^{-3}\right) + \lambda a^5 r^{-3}\sin^2\theta\cos^2\phi\,.$$

This form has the advantage that it is immediately obvious that it matches the boundary condition at $r = a$.