

Physics 712 Chapter 2 Problems

1. [10] A point charge q is at the position $(x, y, z) = (0, 0, h)$ above a grounded conducting plane at $z = 0$. Find the potential everywhere. Find the electric field on the surface $z = 0$, check that it is normal, and find the surface charge density σ on the surface. Integrate the charge density over the entire plane.

Since we have a grounded conducting plane, we add a mirror charge $-q$ at $(0, 0, -h)$.

The potential is then

$$\Phi(x, y, z) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{x^2 + y^2 + (z-h)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+h)^2}} \right]$$

The electric field is minus the gradient of the potential. Evaluating this on the surface $z = 0$ yields

$$\begin{aligned} \mathbf{E}(x, y, 0) &= -\nabla\Phi(x, y, z)|_{z=0} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (z-h)\hat{\mathbf{z}}}{[x^2 + y^2 + (z-h)^2]^{3/2}} - \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (z+h)\hat{\mathbf{z}}}{[x^2 + y^2 + (z+h)^2]^{3/2}} \right\}_{z=0} \\ &= \frac{1}{4\pi\epsilon_0} \frac{-2qh\hat{\mathbf{z}}}{(x^2 + y^2 + h^2)^{3/2}} = \frac{-hq\hat{\mathbf{z}}}{2\pi\epsilon_0(x^2 + y^2 + h^2)^{3/2}} \end{aligned}$$

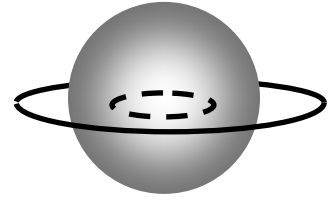
This is obviously perpendicular to the surface. Also, by using the formula $\mathbf{E} = \sigma\hat{\mathbf{n}}/\epsilon_0$ on the surface, we discover

$$\sigma(x, y) = \frac{-hq}{2\pi} (x^2 + y^2 + h^2)^{-3/2}$$

The total charge on the surface is just the integral of this expression over the surface, which can easily be done in polar coordinates, so we have

$$q_{\text{plane}} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \sigma(x, y) = \frac{-hq}{2\pi} \int_0^{2\pi} d\phi \int_0^{\infty} \frac{\rho d\rho}{(\rho^2 + h^2)^{3/2}} = \frac{hq}{\sqrt{\rho^2 + h^2}} \Bigg|_{\rho=0}^{\rho=\infty} = -hqh^{-1} = -q.$$

2. [10] A ring of charge of total charge q forms a circle of radius $2a$ in the xy -plane around a conducting sphere of radius a . Find the potential and electric field everywhere along the z -axis for $z > a$ if the conducting sphere is (a) grounded (b) neutral.



The ring should be broken up into many tiny segments each with charge dq . Each of these, if at a distance r from the center, will make an image charge of magnitude $-(a/r)dq$ at a distance a^2/r from the center. Since all of these charges are at a distance $r = 2a$, the resulting charge will be $-\frac{1}{2}dq$ and will be at a distance $\frac{1}{2}a$. Hence the image charges will form their own ring of radius $\frac{1}{2}a$ with total charge $-\frac{1}{2}q$, as sketched above (dashed ring). For the grounded sphere, that is all we need. A point on the z axis will be at a distance of $\sqrt{z^2 + (2a)^2}$ from the first ring and a distance of $\sqrt{z^2 + (\frac{1}{2}a)^2}$ from the image ring, so the total potential will be

$$\Phi_{\text{grounded}} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{\sqrt{z^2 + 4a^2}} - \frac{\frac{1}{2}}{\sqrt{z^2 + \frac{1}{4}a^2}} \right\} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{\sqrt{z^2 + 4a^2}} - \frac{1}{\sqrt{4z^2 + a^2}} \right\}$$

By symmetry, the electric field will point along the z -axis, and therefore it will be given by

$$\mathbf{E}_{\text{grounded}} = -\hat{\mathbf{z}} \frac{\partial \Phi}{\partial z} = \frac{q\hat{\mathbf{z}}}{4\pi\epsilon_0} \left\{ \frac{z}{(z^2 + 4a^2)^{3/2}} - \frac{4z}{(4z^2 + a^2)^{3/2}} \right\}.$$

For the neutral sphere, one can tell that these are not the correct answer, since the image charge creates an electric field that, when integrated over the surface of the sphere, indicates a total charge of $-\frac{1}{2}q$ on the sphere. We must cancel this. If we add an *additional* fictitious charge of $+\frac{1}{2}q$ at the center, this will create an additional potential that is uniform on the surface of the sphere, so the sphere will still have constant potential (good), but it will also have no net flux leaving the sphere, indicating no net charge on the sphere (also good). Hence the potential and electric field will be

$$\Phi_{\text{neutral}} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{\sqrt{z^2 + 4a^2}} - \frac{1}{\sqrt{4z^2 + a^2}} + \frac{1}{2z} \right\},$$

$$\mathbf{E}_{\text{neutral}} = -\hat{\mathbf{z}} \frac{\partial \Phi}{\partial z} = \frac{q\hat{\mathbf{z}}}{4\pi\epsilon_0} \left\{ \frac{z}{(z^2 + 4a^2)^{3/2}} - \frac{4z}{(4z^2 + a^2)^{3/2}} + \frac{1}{2z^2} \right\}.$$