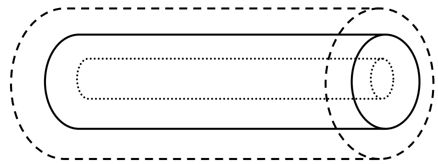
Physics 712
Chapter 1 Problems

1. [10] Consider a long (infinite) cylinder of charge with radius *R*, centered along the *z*-axis, with charge density uniformly spread over its volume, with charge per unit length λ. What is the electric field everywhere?



By symmetry, the field must point away from the cylinder, and depend only on the distance *r* from the center of the cylinder, so $\mathbf{E}(\mathbf{x}) = E(r)\hat{\mathbf{r}}$. For $r > R$, we first draw a Gaussian surface of radius *r* and length *L* that is larger than the charge cylinder (dashed cylinder). The charge inside this cylinder will be λL. There will be no electric field pointing out of the end caps, and the electric field is constant and everywhere normal from the lateral surface of the cylinder. Gauss's Law therefore says

$$\lambda L = \varepsilon_0 \int_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \varepsilon_0 E(r)\int_S da = \varepsilon_0 E(r)\,2\pi r L .$$

Solving, we find $E(r) = \dfrac{\lambda}{2\pi\varepsilon_0 r}$.

For $r < R$, we use a cylinder that is inside the charge cylinder (dotted curve). It will only contain a fraction of the total charge, reduced by the cross-sectional area of the Guassian surface compared to the charge cylinder, so it is reduced by a factor of r^2/R^2. We then proceed exactly as before,

$$\lambda L r^2/R^2 = \varepsilon_0 \int_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = E(r)\int_S da = \varepsilon_0 E(r)\,2\pi r L .$$

Solving, we find $E(r) = \dfrac{\lambda r}{2\pi R^2 \varepsilon_0}$. Combining our formulas, we have

$$\mathbf{E}(\mathbf{x}) = \frac{\lambda}{2\pi\varepsilon_0}\hat{\mathbf{r}}\begin{cases} r^{-1} & \text{if } r > R, \\ rR^{-2} & \text{if } r < R. \end{cases}$$

2. **[10] Consider the electric potential from a neutral hydrogen atom, given in spherical coordinates by**

$$\Phi = \frac{q}{4\pi\varepsilon_0} e^{-2r/a} \left(\frac{1}{r} + \frac{1}{a} \right)$$

where *q* is the fundamental charge, *a* is the Bohr radius. Find the electric field everywhere. Then find the charge density everywhere. Be careful when finding the charge at the origin; you may have to apply Gauss's Law to a small sphere around the origin.

Since the electric field depends only on *r*, we find

$$\mathbf{E} = -\nabla\Phi = -\frac{q\hat{\mathbf{r}}}{4\pi\varepsilon_0} \frac{\partial}{\partial r} \left[e^{-2r/a} \left(\frac{1}{r} + \frac{1}{a} \right) \right] = -\frac{q\hat{\mathbf{r}}}{4\pi\varepsilon_0} \left[-\frac{2}{a} e^{-2r/a} \left(\frac{1}{r} + \frac{1}{a} \right) - e^{-2r/a} \frac{1}{r^2} \right]$$
$$= \frac{q\hat{\mathbf{r}}}{4\pi\varepsilon_0} e^{-2r/a} \left(\frac{1}{r^2} + \frac{2}{ra} + \frac{2}{a^2} \right).$$

We now start to naively calculate the charge density by using $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$, so we have

$$\rho(\mathbf{x}) = \varepsilon_0 \nabla \cdot \mathbf{E} = \varepsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 E_r \right) = \frac{q}{4\pi r^2} \frac{\partial}{\partial r} \left[e^{-2r/a} \left(1 + \frac{2r}{a} + \frac{2r^2}{a^2} \right) \right]$$
$$= \frac{q}{4\pi r^2} \left[-e^{-2r/a} \left(\frac{2}{a} + \frac{4r}{a^2} + \frac{4r^2}{a^3} \right) + e^{-2r/a} \left(\frac{2}{a} + \frac{4r}{a^2} \right) \right] = \frac{-q}{\pi a^2} e^{-2r/a}.$$

This looks straightforward enough, but we have to be careful at the origin, because we are taking the derivative of something that diverges. To understand what is happening, consider the charge inside a sphere of radius *r*. We can find this using Gauss's Law, so we have

$$q(r) = \varepsilon_0 \int_S \hat{\mathbf{n}} \cdot \mathbf{E}(\mathbf{x}) \, da = \frac{q}{4\pi} e^{-2r/a} \left(\frac{1}{r^2} + \frac{2}{ra} + \frac{2}{a^2} \right) \int_S da = \frac{q}{4\pi} e^{-2r/a} \left(\frac{1}{r^2} + \frac{2}{ra} + \frac{2}{a^2} \right) 4\pi r^2$$
$$= q e^{-2r/a} \left(1 + \frac{2r}{a} + \frac{2r^2}{a^2} \right)$$

In the limit of $r \to 0$, this is *q*, which implies that there is a charge of *q* right at the origin, which we previously missed. We can fix this by simply adding a delta-function charge at the origin, so we have

$$\rho(\mathbf{x}) = q\delta^3(\mathbf{x}) - \frac{q}{\pi a^2} e^{-2r/a}.$$