Physics 712
Solutions to Chapter 11 Problems

2. [15] Consider a particle moving along the *x*-axis whose 4-velocity is given at proper time τ by $U^\mu = c(\cosh\phi, \sinh\phi, 0, 0)$, where ϕ is an unknown function of time.

(a) Check that $U \cdot U = c^2$. Find the proper acceleration $a(\tau)$ at time τ for an arbitrary function $\phi(\tau)$.

Trivially, $U \cdot U = c^2\cosh^2\phi - c^2\sinh^2\phi = c^2$. The four-acceleration is given by

$$A^\mu = \frac{d}{d\tau}U^\mu = \left(c\sinh\phi\frac{d\phi}{d\tau}, c\cosh\phi\frac{d\phi}{d\tau}, 0, 0\right)$$

We then have

$$a = \sqrt{-A\cdot A} = \sqrt{-c^2\left(\frac{d\phi}{d\tau}\right)^2\sinh^2\phi + c^2\left(\frac{d\phi}{d\tau}\right)^2\cosh^2\phi} = \sqrt{c^2\left(\frac{d\phi}{d\tau}\right)^2} = c\left|\frac{d\phi}{d\tau}\right|.$$

(b) Suppose $a(\tau) = g$, a constant. Assuming the particle starts at the origin at $\tau = 0$ and is initially at rest, find $\phi(\tau)$, $U(\tau)$ and $x(\tau)$.

Initially at rest means that at $\tau = 0$ we have $U^\mu = (c, 0, 0, 0)$, or $\phi = 0$. Assuming the absolute values is always positive, we now integrate the equation $c(d\phi/d\tau) = g$, and find

$$\phi(\tau) = \frac{g}{c}\tau$$

This can, of course, be substituted into the four velocity to yield

$$U(\tau) = c\left(\cosh(g\tau/c), \sinh(g\tau/c), 0, 0\right)$$

Since $U^\mu = dx^\mu/d\tau$, we can integrate these equations to yield

$$x(\tau) = \frac{c^2}{g}\left(\sinh\left(\frac{g\tau}{c}\right), \cosh\left(\frac{g\tau}{c}\right) - 1, 0, 0\right).$$

The constant of integration was chosen in each case to make sure $x(0) = (0,0,0,0)$.

(c) How much proper time (in years) would it take to get to Alpha Centauri (4.3 *c*·y), the center of our galaxy (2.6×10^4 *c*·y), or the edge of the visible universe (4.5×10^{10} *c*·y) if you start at rest and accelerate in a straight line at proper acceleration *g* = 9.8 m/s^2?

We simply let *x* be the distance to our object and solve for τ in each case.

$$x = \frac{c^2}{g}\left[\cosh\left(g\tau/c\right) - 1\right],$$

$$\cosh\left(\frac{g\tau}{c}\right) = \frac{xg}{c^2} + 1,$$

$$\tau = \frac{c}{g}\cosh^{-1}\left(\frac{xg}{c^2}+1\right) = \frac{2.998\times10^8 \text{ m/s}}{9.8 \text{ m/s}^2}\cosh^{-1}\left(\frac{x}{c}\cdot\frac{9.8 \text{ m/s}^2}{2.998\times10^8 \text{ m/s}}+1\right)$$

$$= \frac{3.059\times10^7 \text{ s}}{3.156\times10^7 \text{ s/yr}}\cosh^{-1}\left(\frac{x}{c}\cdot\frac{3.156\times10^7 \text{ s/yr}}{3.059\times10^7 \text{ s}}+1\right) = \left(0.9693 \text{ yr}\right)\cosh^{-1}\left(\frac{x}{0.9693c\cdot\text{yr}}+1\right).$$

We now simply substitute each of the distances into the formula to get the final answer:

$$\alpha \text{ Centauri:} \quad \tau = \left(0.9693 \text{ yr}\right)\cosh^{-1}\left(\frac{4.3c\cdot\text{yr}}{0.9693c\cdot\text{yr}}+1\right) = 2.305 \text{ yr},$$

$$\text{Center of Galaxy:} \quad \tau = \left(0.9693 \text{ yr}\right)\cosh^{-1}\left(\frac{2.6\times10^4 c\cdot\text{yr}}{0.9693c\cdot\text{yr}}+1\right) = 10.56 \text{ yr},$$

$$\text{Edge of Universe:} \quad \tau = \left(0.9693 \text{ yr}\right)\cosh^{-1}\left(\frac{4.5\times10^{10} c\cdot\text{yr}}{0.9693c\cdot\text{yr}}+1\right) = 24.48 \text{ yr}.$$